A Discontinuous Galerkin Coupled Wave Propagation/Circulation Model

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Abstract On large geographic scales, ocean waves are represented in a spectral sense via the action balance equation, which propagates action density through both geographic and spectral space. In this paper, a new computational spectral wave model is developed by using discontinuous Galerkin (DG) methods in both geographic and spectral space. DG methods allow for the use of unstructured geographic meshes and higher-order approximations for action propagation in both geographic and spectral space, which we show leads to increased accuracy. This DG spectral wave propagation model is verified and validated through comparisons to manufactured and analytic solutions as well as to the Simulating WAves Nearshore (SWAN) model. Coupled wave/circulation models are needed for many applications including for the interaction between waves and currents during daily wind and tide driven flows. We loosely couple the new DG spectral wave model to the DG Shallow Water Equation Model (DG-SWEM), an existing DG circulation model. In addition to formulating the DG method for the coupled wave/circulation model, we derive an a priori error estimate. Preliminary numerical results of the DG coupled wave/circulation model are presented with comparisons to DG-SWEM coupled tightly to SWAN.

Keywords Discontinuous Galerkin methods \cdot Shallow water equations \cdot Action balance equation \cdot Coupled model \cdot A priori error estimate

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Waves and circulation processes interact in daily wind and tide driven flows and in more extreme events such as hurricanes. The winds blowing over the shallow continental shelf create storm surge and currents, which push waves farther inland and/or shift them alongshore. Meanwhile, offshore and local waves dissipate due to changes in bathymetry and bottom friction, and this dissipation creates alongshore currents and set-up. However, waves and currents are modeled separately using different approaches despite their interaction. Long-wave circulation processes, which include currents, tides and storm surge, can be represented by the shallow water equations, which conserve mass and momentum. This approach is impractical for short, wind-driven waves on large geographic scales due to the fine resolution that would be required. Therefore, wind-waves are instead represented in a spectral sense, governed by the action balance equation [13], which propagates action density through both geographic and spectral space. Even though wind-waves and circulation are modeled separately, it is important to account for their interactions by coupling their respective models. In the Gulf of Mexico, the wave-induced set-up can contribute as much as 35 percent of the total water column [10].

SWAN+ADCIRC is a tightly-coupled wave/circulation model that couples the Simulating WAves Nearshore (SWAN) model, a widely-used nearshore wave model [1], and the ADvanced CIRCulation (ADCIRC) model, a widely-used coastal circulation model [20]. Because SWAN has recently been extended to run on unstructured meshes [27], the coupled model runs on the same unstructured mesh. This eliminates interpolation error that occurs when two different meshes are used instead. In addition, unstructured meshes allow for greater flexibility and refinement, which are needed to properly resolve and represent complex coastal regions. The coupled model is run on the same executable and information is passed directly through local memory. SWAN+ADCIRC has successfully been used for hindcasting recent hurricanes in the Gulf of Mexico [7,8,10].

ADCIRC employs a continuous-Galerkin (CG) finite-element method, which is limited to linear approximations, has difficulties handling strong advection and is not locally mass conservative [4]. Recent efforts have extended ADCIRC to use a discontinuous-Galerkin (DG) method; this DG Shallow Water Equation Model (DG-SWEM) can employ higher order approximations and can handle advection dominated flows [16]. In addition, DG models are locally and globally conservative, which can be important when coupling to transport equations [3,6]. DG-SWEM is mature, so a comparison to ADCIRC(CG) is not considered herein. In this work, DG-SWEM has been coupled tightly with SWAN, in a manner similar to [10], which allows both models to run as the same executable program, and on the same unstructured mesh. However, the finite-difference solution method in SWAN requires information at the vertices, so high-order information from DG-SWEM is lost in the coupling.

The unstructured-mesh version of SWAN is one example of the shift within the spectral wave modeling community toward unstructured meshes, either by adapting finite difference methods (as in SWAN [27]) or implementing finite volumes [23] or finite elements [14]. However, all these models still employ finite differences in spectral space. Another model utilizes DG in geographic space with a collocation method in spectral space [26]. In contrast to these existing models, we investigate DG methods in both geographic and spectral space for our spectral wave model. Using DG methods allows for easy implementation of higher-order approximations in both geographic and spectral space. We show that utilizing higher-order approximations, particularly in spectral space, leads to higher accuracy. In addition, we find it can be more beneficial to increase the order of approximation than refine the mesh. This could be particularly beneficial when refinement is not an option to obtain increased accuracy

due to specific mesh requirements for a source term formulation. The resulting coupled DG wave/circulation model executes on the same unstructured mesh, which eliminates interpolation error between meshes as does the previously coupled SWAN+ADCIRC model [10]. In addition, higher-order information can be passed between the models.

Thus, DG-SWEM is coupled tightly with SWAN, or loosely with a new DG wave model. The main focus of this paper is to formulate the DG method for the coupled wave propagation/circulation model and perform an a priori error estimate. Although this DG formulation of the shallow water equations has been analyzed in [5], no coupled wave/circulation model has been analyzed for any numerical scheme. We also present verification and validation of the DG spectral wave model including comparisons to SWAN and preliminary numerical results of the coupled DG wave/circulation model comparing to DG-SWEM coupled to SWAN.

This paper is organized as follows. In the following section we introduce the governing equations of the coupled wave/circulation model as well as some necessary notations and definitions. We continue in Sect. 3 by formulating the weak form, defining the approximation spaces, and formulating the discrete weak form of the coupled model. In Sect. 4, we form the error equations which are necessary for performing the a priori error estimate presented in Sect. 5. We continue in Sect. 6 with verification and validation of the DG spectral wave model and preliminary results of the coupled wave/circulation models. Concluding remarks are in Sect. 7.

2 Governing Equations

The coupled wave/circulation model consists of the shallow water equations, which describe the circulation processes, and the action balance equation, which describes waves in a statistical sense. We simplify the error analysis that follows by only considering the geographic domain to be one-dimensional; however, all arguments in the analysis are valid in twodimensions and the numerical results presented in Sect. 6 are also in two-dimensions. We consider the coupled model on the domain Ω , which is a tensor product of the geographic domain, $\Xi \in \mathbb{R}$, and the spectral domain, $\kappa = (\sigma, \theta) \in \mathbb{R}^2$ where $\sigma, 0 < \sigma \leq \sigma_{\text{max}}$, is the relative frequency, which is the frequency observed in a frame of reference moving with the current, and θ is the wave direction, so $\Omega = \Xi \times \kappa \in \mathbb{R} \times \mathbb{R}^2$, for time t > 0. We denote the boundary of the domain as $\partial \Omega$ with outward normal $\mathbf{n} = (n_x, n_\sigma, n_\theta)$ and denote the boundary of the geographic domain as $\partial \Xi$ with outward normal n_x . The one-dimensional coupled wave/circulation system for which we perform the error analysis consists of the continuity equation [5]

$$\xi_t + \nabla_x \cdot (uH) = 0, \tag{1}$$

the non-conservative form of the momentum equation

$$u_t + u \cdot \nabla_x u + g \nabla_x \xi - \mu \Delta u = F_x, \tag{2}$$

and the action balance equation [1]

$$N_t + \nabla \cdot \mathbf{c}N = \frac{S}{\sigma},\tag{3}$$

where $\nabla_* = \frac{\partial}{\partial *}$ and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial \sigma}, \frac{\partial}{\partial \theta}\right)$. The unknown variables are ξ , the elevation of the free water surface from the geoid; u, the depth-integrated velocity; and N, the action density. $H = \xi + h_b$ is the total height of the water column where h_b is the depth of the water below

the geoid (bathymetry); g is the gravitational acceleration; $\mu > 0$ is the eddy viscosity; and F_x is the forcing function. The forcing term includes the wave radiation stress gradient, the Coriolis effect, tidal potential and bottom friction. Here for simplicity in the analysis, the only forcing considered is due to the wave radiation stress gradient; that is

$$F_x = \frac{\tau_{sx,wave}}{\rho H}$$

where ρ is the density of the water. The radiation stress gradient is [19]

$$\tau_{sx,wave} = -\nabla_x S_{xx},$$

where

$$S_{xx} = \rho g \iint \left(\left(n \cos^2 \theta + n - \frac{1}{2} \right) \sigma N \right) d\sigma d\theta,$$

and $n = \frac{1}{2} \left(1 + \frac{2kH}{\sinh(2kH)} \right)$. The action balance equation represents the rate of change of action density in time, the propagation of action density through geographic space with propagation velocity c_x and c_y , and the propagation of action density through spectral space, which represents frequency shifting due to changes in depth and current with propagation velocity c_{σ} and depth- and current-induced refraction with propagation velocity c_{θ} . The propagation velocities $\mathbf{c} = (c_x, c_{\sigma}, c_{\theta})$ in their simplified, one geographic dimensional form are [13]

$$c_x = c_g \cos(\theta) + u, \tag{4}$$

$$c_{\sigma} = \frac{k\sigma}{\sinh(2kH)} \left(H_t + u\nabla_x H\right) - c_g k \nabla_x u \cos^2 \theta, \tag{5}$$

$$c_{\theta} = \frac{\sigma}{\sinh(2kH)} \nabla_{x} H \sin\theta + \nabla_{x} u \cos\theta \sin\theta, \qquad (6)$$

where $c_g = nc$ is the group velocity, $c = \sigma/k$ is the phase speed, and k is the wave number, which is related to the frequency via the dispersion relationship $\sigma^2 = gk \tanh(kH)$. The source term, S, accounts for wind input, S_{in} , dissipation, S_d , and nonlinear wave-wave interactions, S_{nl} . We omit the details of the source term, but note that it can be written as

$$\frac{S}{\sigma} = S_{in} + S_d + S_{nl} = (f_S(H) + g_S(N))N$$

where f_S is Lipschitz continuous and g_S , which can depend on integral values of the action density, is also Lipschitz continuous. For the full two-dimensional system of equations, derivatives over geographic space are taken over both x and y and the full two-dimensional radiation stress tensor is used.

For the error analysis only, we will consider the deep water assumption of $H \gg 0$ and thus $\tanh(kH) \approx 1$. This deep water assumption allows for the simplification of the group velocity as $c_g = g/\sigma$, n = 1/2, and the dispersion relationship becomes $\sigma = \sqrt{gk}$. When the deep water assumption is valid, the group velocity no longer depends on the depth of the water. To explore this, we plot in Fig. 1 the group velocity for a range of depths for several different frequencies. When the frequency is 0.03 Hz, the group velocity slows its dependence on depth around H = 800 m. However, for the higher frequencies of 0.1 and 1.0 Hz we see that the group velocity no longer depends on depth for depths larger than 155 and 1.6 m, respectively. Therefore if lower frequencies are included in the domain then we must restrict ourselves in the analysis to water depths larger than 800 m for the deep water assumption to be valid, but we can consider shallower water if we only consider higher frequencies. We



Fig. 1 The group velocity for different depths for several frequencies

also need for the horizontal length scale to be much larger than the vertical length scale, so that the shallow water equations are valid. For the error analysis, we rewrite the propagation velocities using the continuity equation in the expression for c_{σ} and by approximating the derivative of the total water depth by the derivative of the bathymetry in the expression for c_{θ} and obtain

$$c_{\sigma} = \frac{k\sigma}{\sinh(2kH)} \left(-H\nabla_{x}u\right) - c_{g}k\nabla_{x}u\cos^{2}\theta,$$
(7)

$$c_{\theta} = \frac{\sigma}{\sinh(2kH)} \nabla_x h_b \sin\theta + \nabla_x u \cos\theta \sin\theta.$$
(8)

For the analysis, we also approximate the total water depth by the bathymetry in the forcing function so that

$$F_x = \frac{\tau_{sx,wave}}{\rho h_b}.$$
(9)

For the shallow water equations, we consider the geographic boundary to be divided into an inflow and outflow region $\partial \Xi = \partial \Xi_{in} \cup \partial \Xi_{out}$, where

 $\partial \Xi_{in} = \{ x \in \partial \Xi : u \cdot n_x < 0 \},\\ \partial \Xi_{out} = \{ x \in \partial \Xi : u \cdot n_x \ge 0 \},$

with the following boundary conditions

$$u(x,t) = \widehat{u}(x,t), \quad \partial \Xi,$$

$$\xi(x,t) = \widehat{\xi}(x,t), \quad \partial \Xi_{in}$$

For the action balance equation, we consider the entire boundary to be divided into an inflow and outflow region $\partial \Omega = \partial \Omega_{in} \cup \partial \Omega_{out}$, where

$$\partial \Omega_{in} = \{ (x, \sigma, \theta) \in \partial \Omega : \mathbf{c} \cdot \mathbf{n} < 0 \},\\ \partial \Omega_{out} = \{ (x, \sigma, \theta) \in \partial \Omega : \mathbf{c} \cdot \mathbf{n} \ge 0 \},$$

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with the following boundary condition

$$N(x, \sigma, \theta, t) = N(x, \sigma, \theta, t), \quad \partial \Omega_{in}.$$

2.1 Notation and Definitions

Let $\{\mathscr{T}_h\}_{h>0}$ be a family of regular finite element partitions of Ω such that no single element $\Omega_e = \Xi_g \times \kappa_s$ crosses the boundary $\partial \Omega$ and \mathscr{T}_h is locally quasi-uniform [2]. We assume each element Ω_e and Ξ_g is Lipschitz and affinely equivalent to a reference element [2]. Let h_g be the diameter of an element Ξ_g , h_e be the diameter of an element Ω_e , and h be the maximum element diameter.

For any $v \in H^1(\Xi_g)$, for each element Ξ_g , we denote the trace v^{\pm} on the interior faces of Ξ_g , γ_i , by

$$v^{-}(x) = \lim_{s \to 0^{-}} v(x + sn_i), \quad v^{+}(x) = \lim_{s \to 0^{+}} v(x + sn_i),$$

where $x \in \gamma_i$ and n_i is a fixed unit vector normal to γ_i . Similarly for any $w \in H^1(\Omega_e)$, for each element Ω_e , we denote the trace w^{\pm} on the interior edges of Ω_e , λ_j , by

$$w^{-}(\mathbf{x}) = \lim_{s \to 0^{-}} w(\mathbf{x} + s\mathbf{n}_j), \quad w^{+}(\mathbf{x}) = \lim_{s \to 0^{+}} w(\mathbf{x} + s\mathbf{n}_j),$$

where $\mathbf{x} = (x, \sigma, \theta) \in \lambda_j$ and \mathbf{n}_j is a fixed unit vector normal to λ_j . We then define the average and jump of a function v over a geographic element face γ_i as

$$\{v\} = \frac{1}{2}(v^+ + v^-), \quad [v] = v^- - v^+,$$

and respectively, the average and jump of a function w over an element edge λ_i as

$$\{w\} = \frac{1}{2}(w^+ + w^-), \quad [w] = w^- - w^+.$$

We denote the sum over all elements in Ξ as $\sum_{\Xi_g \subset \Xi}$, the sum over all elements in the entire domain as $\sum_{\Omega_e \subset \Omega}$, the sum over all the geographic interior edges as $\sum_{\gamma_i \in \Gamma_{\Xi}}$, and the sum over all interior faces as $\sum_{\lambda_i \in \Gamma_{\Omega}}$.

We use the $L^2(R)$ inner product notation $(\cdot, \cdot)_R$ for the interior of a domain $R \in \mathbb{R}^d$, d = 1, 2, 3, and the notation $\langle \cdot, \cdot \rangle$ for the inner product over the edges or faces. Denote by $|| \cdot ||_R$ the L^2 norm on region R and note that for a function $f \in L^2(\Xi_g)$ or $g \in L^2(\Omega_e)$ that

$$||f||_{\mathcal{Z}} = \sum_{\mathcal{Z}_g \subset \mathcal{Z}} ||f||_{\mathcal{Z}_g}, \quad ||g||_{\Omega} = \sum_{\Omega_e \subset \Omega} ||g||_{\Omega_e}.$$

Let norms in other Sobolev spaces W(R) be denoted $|| \cdot ||_{W(R)}$ and for a time dependent function f = f(x, t)

$$||f||_{L^{\infty}(0,T;W(R))} = \operatorname{ess\,sup}_{0 \le t \le T} ||f(\cdot,t)||_{W(R)}.$$

3 Weak Formulations

Both the continuity equation and action balance equation will be discretized via a discontinuous Galerkin (DG) method. The momentum equation will be discretized via the nonsymmetric interior penalty Galerkin (NIPG) method with the upwinding technique of Lesaint and Raviart [18] applied to the advection term. We proceed by multiplying the continuity and momentum equations by test functions $v \in H^1(\Xi_g)$ and $v \in H^1(\Xi_g)$ on each geographic element $\Xi_g \subset \Xi$, integrate by parts and sum over each equation's results to obtain the weak formulations. Similarly, we multiply the action balance equation by a test function $w \in H^1(\Omega_e)$ on each element $\Omega_e \subset \Omega$, integrate by parts and sum over the results to obtain the weak formulation. The system then contains the weak form of the continuity equation

$$\begin{aligned} (\xi_t, \nu)_{\varXi} &- (uH, \nabla_x \nu)_{\varXi} + \sum_{\gamma_i \in \Gamma_{\varXi}} \langle Hu \cdot n_i, [\nu] \rangle_{\gamma_i} + \langle H\widehat{u} \cdot n_x, \nu \rangle_{\partial \varXi_{out}} \\ &= - \langle \widehat{H}\widehat{u} \cdot n_x, \nu \rangle_{\partial \varXi_{in}}, \end{aligned}$$
(10)

where $\widehat{H} = \widehat{\xi} + h$, the momentum equation

$$(u_{l}, v)_{\Xi} + (u \cdot \nabla_{x} u, v)_{\Xi} + (g \nabla_{x} \xi, v)_{\Xi} + \mu (\nabla_{x} u, \nabla_{x} v)_{\Xi} - \sum_{\gamma_{i} \in \Gamma_{\Xi}} \mu \langle \nabla_{x} u \cdot n_{i}, [v] \rangle_{\gamma_{i}} - \mu \langle \nabla_{x} u \cdot n_{x}, v \rangle_{\partial \Xi} = (F_{x}, v)_{\Xi},$$
(11)

and the action balance equation

$$(N_{t}, w)_{\Omega} - (\mathbf{c}N, \nabla w)_{\Omega} + \sum_{\lambda_{j} \in \Gamma_{\Omega}} \langle \mathbf{c}N \cdot \mathbf{n}_{j}, [w] \rangle_{\lambda_{j}} + \langle \mathbf{c}N \cdot \mathbf{n}, w \rangle_{\partial \Omega_{out}} = \left(\frac{S}{\sigma}, w\right)_{\Omega} - \langle \mathbf{c}\widehat{N} \cdot \mathbf{n}, w \rangle_{\partial \Omega_{in}}.$$
(12)

3.1 Approximation Spaces

We define the following two approximation spaces that allow for discontinuities along the edges of elements

$$V_h = \{ v \in L^2(\Xi) : v |_{\mathcal{Z}_g} \in P^p(\mathcal{Z}_g) \quad \forall \mathcal{Z}_g \},$$

$$W_h = \{ w \in L^2(\Omega) : w |_{\Omega_e} \in P^p(\mathcal{Z}_g) * P^q(\kappa_s) \quad \forall \Omega_e \},$$

where P^p is the space of polynomials of degree p defined on the geographic element Ξ_g and P^q is the space of polynomials of degree q defined on the spectral element κ_s .

3.2 Discrete Weak Formulations

We begin by approximating the initial conditions with L^2 projections, computing $\xi_h(\cdot, 0)$, $u_h(\cdot, 0) \in V_h(\Xi)$ and $N_h(\cdot, 0) \in W_h(\Omega)$ to satisfy

$$\begin{aligned} &(\xi_0 - \xi_h(\cdot, 0), v) = 0, \quad v \in V_h(\Xi), \\ &(u_0 - u_h(\cdot, 0), v) = 0, \quad v \in V_h(\Xi), \\ &(N_0 - N_h(\cdot, 0), w) = 0, \quad w \in W_h(\Omega) \end{aligned}$$

We must slightly alter the definition of $\partial \Xi_{in}$ and $\partial \Xi_{out}$ to correspond with $u_h \cdot n_x < 0$ and $u_h \cdot n_x \ge 0$ and the definitions of $\partial \Omega_{in}$ and $\partial \Omega_{out}$ to correspond with $\mathbf{c}_h \cdot \mathbf{n} < 0$ and $\mathbf{c}_h \cdot \mathbf{n} \ge 0$. The free surface elevation ξ is approximated by $\xi_h \in V_h$ which satisfies the discrete weak form of the continuity equation

$$((\xi_h)_t, \nu)_{\varXi} - (u_h H_h, \nabla_x \nu)_{\varXi} + \sum_{\gamma_i \in \Gamma_{\varXi}} \langle H_h^{\uparrow} \{u_h\} \cdot n_i, [\nu] \rangle_{\gamma_i} + \langle H_h u_h \cdot n_x, \nu \rangle_{\partial \varXi_{out}} = - \langle \widehat{H} u_h \cdot n_x, \nu \rangle_{\partial \varXi_{in}},$$
(13)

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where $H_h = \xi_h + h_b$ and the upwind value of H_h on each geographic interior edge γ_i is defined as

$$H_{h}^{\uparrow} = \begin{cases} H_{h}^{-} \text{ if } \{u_{h}\} \cdot n_{i} > 0, \\ H_{h}^{+} \text{ if } \{u_{h}\} \cdot n_{i} \le 0. \end{cases}$$
(14)

For the momentum equation, we add three stability terms involving $[u_h]$, $[\xi_h]$ and $[S_{xx,h}]$ which are zero for the true solutions. The depth-integrated velocity u is approximated by $u_h \in V_h$ which satisfies the discrete weak form

$$((u_{h})_{t}, v)_{\Xi} + (u_{h} \cdot \nabla_{x} u_{h}, v)_{\Xi} + \sum_{g} \langle |\{u_{h}\} \cdot n_{e}|(u_{h}^{int} - u_{h}^{ext}), v^{int} \rangle_{\partial \Xi_{g}^{-}}$$

$$+ (g \nabla_{x} \xi_{h}, v)_{\Xi} - \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle g[\xi_{h}], \{v\} \cdot n_{i} \rangle_{\gamma_{i}} - \langle g(\xi_{h} - \widehat{\xi}), v \cdot n_{x} \rangle_{\partial \Xi_{in}}$$

$$+ \mu (\nabla_{x} u_{h}, \nabla_{x} v)_{\Xi} - \sum_{\gamma_{i} \in \Gamma_{\Xi}} \mu \langle \{\nabla_{x} u_{h}\} \cdot n_{i}, [v] \rangle_{\gamma_{i}}$$

$$+ \sum_{\gamma_{i} \in \Gamma_{\Xi}} \mu \langle \{\nabla_{x} v\} \cdot n_{i}, [u_{h}] \rangle_{\gamma_{i}} + \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle \alpha[u_{h}], [v] \rangle_{\gamma_{i}}$$

$$- \mu \langle \nabla u_{h} \cdot n_{x}, v \rangle_{\partial \Xi} - \langle \alpha(u_{h} - \widehat{u}), v \rangle_{\partial \Xi}$$

$$+ \mu \langle \nabla v \cdot n_{x}, u_{h} - \widehat{u} \rangle_{\partial \Xi} = (F_{x,h}, v) + \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle [S_{xx,h}], v \rangle_{\gamma_{i}},$$
(15)

where α is a positive parameter which will be further discussed later. $\partial \Xi_g^- = \{x \in \partial \Xi_g : \{u_h\} \cdot n_x < 0\}, v^{int}$ and v^{ext} are the traces of v from the interior and exterior of $\partial \Xi_g$, and when the edge of element Ξ_g belongs to the geographic boundary $\partial \Xi$, the exterior trace value $u_h^{ext} = \hat{u}$. The action density N is approximated by $N_h \in W_h$, which fulfills the discrete weak form of the action balance equation

$$((N_{h})_{t}, w)_{\Omega} - (\mathbf{c}_{h}N_{h}, \nabla w)_{\Omega} + \sum_{\lambda_{j} \in \Gamma_{\Omega}} \langle N_{h}^{\uparrow} \{\mathbf{c}_{h}\} \cdot n_{j}, [w] \rangle_{\lambda_{j}} + \langle \mathbf{c}_{h}N_{h} \cdot n, w \rangle_{\partial \Omega_{out}} = \left(\frac{S_{h}}{\sigma}, w_{h}\right)_{\Omega} - \langle \mathbf{c}_{h}\widehat{N} \cdot n, w \rangle_{\partial \Omega_{in}},$$
(16)

where the upwind value of N_h on each interior edge λ_i is defined as

$$N_h^{\uparrow} = \begin{cases} N_h^- \text{ if } \{\mathbf{c}_h\} \cdot \mathbf{n}_j > 0, \\ N_h^+ \text{ if } \{\mathbf{c}_h\} \cdot \mathbf{n}_j \le 0. \end{cases}$$
(17)

4 Error Equations

We now perform an a priori error analysis of the coupled system (1)–(3). To form the error equations we first must define the L^2 projections ξ_h , \tilde{u}_h and \tilde{N}_h into the approximation spaces for ξ_h , u_h and N_h such that for time $t \ge 0$

$$\begin{split} & \left((\xi_h - \xi)(\cdot, t), v \right) = 0 \quad v \in V_h, \\ & \left((\tilde{u}_h - u)(\cdot, t), v \right) = 0 \quad v \in V_h, \\ & \left((\tilde{N}_h - N)(\cdot, t), w \right) = 0 \quad v \in W_h. \end{split}$$

Define

$$e_{\xi} = \xi_h - \tilde{\xi}_h, \quad \Theta_{\xi} = \xi - \tilde{\xi}_h$$

with similar definitions for e_u , Θ_u , e_N and Θ_N . Let $\tilde{H}_h = \tilde{\xi}_h + h_b$ and note $e_{\xi} = H_h - \tilde{H}_h$.

We now perform manipulations to the weak forms to obtain the error equations, many of these arguments are repeated from Dawson and Proft [5]. We begin with the continuity equation by subtracting the weak form (10) from the discrete weak form (13) and setting $v = e_{\xi}$ to obtain

$$((e_{\xi})_{l}, e_{\xi})_{\Xi} - (u_{h}e_{\xi}, \nabla_{x}e_{\xi})_{\Xi} + \sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle e_{\xi}^{\uparrow}\{u_{h}\} \cdot n_{i}, [e_{\xi}] \rangle_{\gamma_{i}} + \langle e_{\xi}u_{h} \cdot n_{x}, e_{\xi} \rangle_{\partial\Xi_{out}}$$

$$= ((\Theta_{\xi})_{l}, e_{\xi})_{\Xi} - (uH - u_{h}\widetilde{H}_{h}, \nabla_{x}e_{\xi})_{\Xi} + \sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle (uH - \{u_{h}\}\widetilde{H}_{h}^{\uparrow}) \cdot n_{i}, [e_{\xi}] \rangle_{\gamma_{i}}$$

$$+ \langle (\widehat{u}H - u_{h}\widetilde{H}_{h}) \cdot n_{x}, e_{\xi} \rangle_{\partial\Xi_{out}} + \langle \widehat{u}\widehat{H} - u_{h}\widehat{H}) \cdot n_{x}, e_{\xi} \rangle_{\partial\Xi_{in}}.$$
(18)

We then integrate by parts

$$\begin{aligned} -(u_{h}e_{\xi},\nabla_{x}e_{\xi})_{\Xi} + \sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle e_{\xi}^{\uparrow}\{u_{h}\}\cdot n_{i}, [e_{\xi}]\rangle_{\gamma_{i}} + \langle e_{\xi}u_{h}\cdot n_{x}, e_{\xi}\rangle_{\partial\Xi_{out}} \\ &= \frac{1}{2} (\nabla_{x}\cdot u_{h}, e_{\xi}^{2})_{\Xi} - \frac{1}{2} \sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle [u_{h}e_{\xi}^{2}\cdot n_{i}], 1\rangle_{\gamma_{i}} \\ &+ \sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle e_{\xi}^{\uparrow}\{u_{h}\}\cdot n_{i}, [e_{\xi}]\rangle_{\gamma_{i}} + \frac{1}{2} \langle |u_{h}\cdot n_{x}|, e_{\xi}^{2}\rangle_{\partial\Xi_{in}} + \frac{1}{2} \langle |u_{h}\cdot n_{x}|, e_{\xi}^{2}\rangle_{\partial\Xi_{out}}. \end{aligned}$$

Use the fact that $[ab] = \{a\}[b] + [a]\{b\}, \frac{1}{2}[a^2] = [a]\{a\}$ and the definition of e_{ξ}^{\uparrow} to get

$$\begin{split} &-\frac{1}{2}\sum_{\gamma_i\in\Gamma_{\Xi}}\langle [u_he_{\xi}^2\cdot n_i],1\rangle_{\gamma_i}+\sum_{\gamma_i\in\Gamma_{\Xi}}\langle e_{\xi}^{\uparrow}\{u_h\}\cdot n_i,[e_{\xi}]\rangle_{\gamma_i}\\ &=\sum_{\gamma_i\in\Gamma_{\Xi}}\langle e_{\xi}^{\uparrow}[e_{\xi}]\{u_h\}\cdot n_i-\frac{1}{2}[e_{\xi}^2]\{u_h\}\cdot n_i-\frac{1}{2}\{e_{\xi}^2\}[u_h]\cdot n_i,1\rangle_{\gamma_i}\\ &=\sum_{\gamma_i\in\Gamma_{\Xi}}\langle (e_{\xi}^{\uparrow}-\{e_{\xi}\})\{u_h\}\cdot n_i,[e_{\xi}]\rangle_{\gamma_i}-\frac{1}{2}\sum_{\gamma_i\in\Gamma_{\Xi}}\langle \{e_{\xi}^2\},[u_h]\cdot n_i\rangle_{\gamma_i}\\ &=\frac{1}{2}\sum_{\gamma_i\in\Gamma_{\Xi}}\langle |\{u_h\}\cdot n_i|,[e_{\xi}]^2\rangle_{\gamma_i}-\frac{1}{2}\sum_{\gamma_i\in\Gamma_{\Xi}}\langle \{e_{\xi}^2\},[u_h]\cdot n_i\rangle_{\gamma_i}. \end{split}$$

Then integrating $-(uH - u_h \widetilde{H}_h, \nabla_x e_{\xi})_{\Xi}$ by parts we have

$$\begin{aligned} -(uH - u_{h}\widetilde{H}_{h}, \nabla_{x}e_{\xi})_{\Xi} + \langle (\widehat{u}\widehat{H} - u_{h}\widehat{H}) \cdot n_{x}, e_{\xi} \rangle_{\partial \Xi_{in}} + \langle (\widehat{u}H - u_{h}\widetilde{H}_{h}) \cdot n_{x}, e_{\xi} \rangle_{\partial \Xi_{out}} \\ &= (\nabla_{x} \cdot (uH - u_{h}\widetilde{H}_{h}), e_{\xi})_{\Xi} \\ &- \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle [e_{\xi}(uH - u_{h}\widetilde{H}_{h}) \cdot n_{i}], 1 \rangle_{\gamma_{i}} + \langle (u_{h}\widetilde{H}_{h} - u_{h}\widehat{H}) \cdot n_{x}, e_{\xi} \rangle_{\partial \Xi_{in}}. \end{aligned}$$

The continuity error equation is then

$$((e_{\xi})_{t}, e_{\xi})_{\Xi} + \frac{1}{2} (\nabla_{x} \cdot u_{h}, e_{\xi}^{2})_{\Xi} + \frac{1}{2} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle |\{u_{h}\} \cdot n_{i}|, [e_{\xi}]^{2} \rangle_{\gamma_{i}}$$

$$- \frac{1}{2} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle \{e_{\xi}^{2}\}, [u_{h}] \cdot n_{i} \rangle_{\gamma_{i}} + \frac{1}{2} \langle |u_{h} \cdot n|, e_{\xi}^{2} \rangle_{\partial \Xi_{in}} + \frac{1}{2} \langle |u_{h} \cdot n|, e_{\xi}^{2} \rangle_{\partial \Xi_{out}}$$

$$= ((\Theta_{\xi})_{t}, e_{\xi})_{\Xi} + (\nabla_{x} \cdot (uH - u_{h}\widetilde{H}_{h}), e_{\xi})_{\Xi} + \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle (uH - \{u_{h}\}\widetilde{H}_{h}^{\uparrow}) \cdot n_{i}, [e_{\xi}] \rangle_{\gamma_{i}}$$

$$- \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle [e_{\xi}(uH - u_{h}\widetilde{H}_{h}) \cdot n_{i}], 1 \rangle_{\gamma_{i}} + \langle (u_{h}\widetilde{H}_{h} - u_{h}\widehat{H}) \cdot n_{x}, e_{\xi} \rangle_{\partial \Xi_{in}}.$$
(19)

We proceed with the momentum equation, subtract the weak form (11) from the discrete weak form (15) and let $v = e_u$ to obtain

$$((e_{u})_{l}, e_{u})_{\Xi} + (g\nabla_{x}e_{\xi}, e_{u})_{\Xi} - \sum_{\gamma_{l}\in\Gamma_{\Xi}} \langle g[e_{\xi}], \{e_{u}\} \cdot n_{i} \rangle_{\gamma_{l}} - \langle ge_{\xi}, e_{u} \cdot n_{x} \rangle_{\partial\Xi_{in}}$$

$$+ \mu (\nabla_{x}e_{u}, \nabla_{x}e_{u})_{\Xi} + \sum_{\gamma_{l}\in\Gamma_{\Xi}} \langle \alpha[e_{u}], [e_{u}] \rangle_{\gamma_{l}} + \langle \alpha e_{u}, e_{u} \rangle_{\partial\Xi}$$

$$= ((\Theta_{u})_{l}, e_{u})_{\Xi} + (u \cdot \nabla_{x}u - u_{h} \cdot \nabla_{x}u_{h}, e_{u})_{\Xi}$$

$$- \sum_{g} \langle |\{u_{h}\} \cdot n_{e}|(u_{h}^{int} - u_{h}^{ext}), e_{u}^{int} \rangle_{\partial\Xi_{g}^{-}} + (g\nabla_{x}\Theta_{\xi}, e_{u})_{\Xi}$$

$$- \sum_{\gamma_{l}\in\Gamma_{\Xi}} \langle g[\Theta_{\xi}], \{e_{u}\} \cdot n_{l} \rangle_{\gamma_{l}} - \langle g\Theta_{\xi}, e_{u} \cdot n_{x} \rangle_{\partial\Xi_{in}} + \mu (\nabla_{x}\Theta_{u}, \nabla_{x}e_{u})_{\Xi}$$

$$+ \sum_{\gamma_{l}\in\Gamma_{\Xi}} \langle \alpha[\Theta_{u}], [e_{u}] \rangle_{\gamma_{l}} - \sum_{\gamma_{l}\in\Gamma_{\Xi}} \langle \mu\{\nabla_{x}\Theta_{u}\} \cdot n_{l}, [e_{u}] \rangle_{\gamma_{l}}$$

$$+ \sum_{\gamma_{l}\in\Gamma_{\Xi}} \langle \mu\{\nabla_{x}e_{u}\} \cdot n_{l}, [\Theta_{u}] \rangle_{\gamma_{l}} - \mu \langle \nabla_{x}\Theta_{u} \cdot n_{x}, e_{u} \rangle_{\partial\Xi} + \mu \langle \nabla_{x}e_{u} \cdot n_{x}, \Theta_{u} \rangle_{\partial\Xi}$$

$$+ \langle \alpha\Theta_{u}, e_{u} \rangle_{\partial\Xi} + (F_{x,h} - F_{x}, e_{u})_{\Xi} + \sum_{\gamma_{l}\in\Gamma_{\Xi}} \langle [S_{xx,h} - S_{xx}], \{e_{u}\} \rangle_{\gamma_{l}}.$$
(20)

Integrating $(g\nabla_x e_{\xi}, e_u)_{\Xi}$ by parts we get

$$(g\nabla_{x}e_{\xi}, e_{u})_{\Xi} - \sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle g[e_{\xi}], \{e_{u}\}\cdot n_{i}\rangle_{\gamma_{i}} - \langle ge_{\xi}, e_{u}\cdot n_{x}\rangle_{\partial\Xi_{in}}$$

= $-(ge_{\xi}, \nabla_{x}\cdot e_{u})_{\Xi} + \sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle g\{e_{\xi}\}, [e_{u}]\cdot n_{i}\rangle_{\gamma_{i}} + \langle ge_{\xi}, e_{u}\cdot n_{x}\rangle_{\partial\Xi_{out}}.$

The momentum error equation then becomes

$$\begin{aligned} &((e_u)_t, e_u)_{\varXi} - (ge_{\xi}, \nabla_x \cdot e_u)_{\varXi} + \sum_{\gamma_i \in \Gamma_{\varXi}} \langle g\{e_{\xi}\}, [e_u] \cdot n_i \rangle_{\gamma_i} + \langle ge_{\xi}, e_u \cdot n_x \rangle_{\partial \varXi_{out}} \\ &+ \mu (\nabla_x e_u, \nabla_x e_u)_{\varXi} + \sum_{\gamma_i \in \Gamma_{\varXi}} \langle \alpha[e_u], [e_u] \rangle_{\gamma_i} + \langle \alpha e_u, e_u \rangle_{\partial \varXi} \\ &= ((\Theta_u)_t, e_u)_{\varXi} + (u \cdot \nabla_x u - u_h \cdot \nabla_x u_h, e_u)_{\varXi} \\ &- \sum_g \langle |\{u_h\} \cdot n_e| (u_h^{int} - u_h^{ext}), e_u^{int} \rangle_{\partial \varXi_g}^- + (g \nabla_x \Theta_{\xi}, e_u)_{\varXi} \end{aligned}$$

$$-\sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle g[\Theta_{\xi}], \{e_{u}\}\cdot n_{i}\rangle_{\gamma_{i}} - \langle g\Theta_{\xi}, e_{u}\cdot n_{x}\rangle_{\partial\Xi_{in}} + \mu(\nabla_{x}\Theta_{u}, \nabla_{x}e_{u})_{\Xi}$$

$$+\sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle \alpha[\Theta_{u}], [e_{u}]\rangle_{\gamma_{i}} - \sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle \mu\{\nabla_{x}\Theta_{u}\}\cdot n_{i}, [e_{u}]\rangle_{\gamma_{i}}$$

$$+\sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle \mu\{\nabla_{x}e_{u}\}\cdot n_{i}, [\Theta_{u}]\rangle_{\gamma_{i}} - \mu\langle\nabla_{x}\Theta_{u}\cdot n_{x}, e_{u}\rangle_{\partial\Xi} + \mu\langle\nabla_{x}e_{u}\cdot n, \Theta_{u}\rangle_{\partial\Xi}$$

$$+ \langle \alpha\Theta_{u}, e_{u}\rangle_{\partial\Xi} + (F_{x,h} - F_{x}, e_{u})_{\Xi} + \sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle [S_{xx,h} - S_{xx}], \{e_{u}\}\rangle_{\gamma_{i}}.$$
(21)

Finally, we note that the action balance equation contains analogous terms to the continuity equation. Therefore, performing the same manipulations as before, we obtain action balance error equation

$$((e_{N})_{t}, e_{N})_{\Omega} + \frac{1}{2} \left(\nabla \cdot \mathbf{c}_{h}, e_{N}^{2} \right)_{\Omega} + \frac{1}{2} \sum_{\lambda_{j} \in \Gamma_{\Omega}} \langle |\{\mathbf{c}_{h}\} \cdot \mathbf{n}|, [e_{N}]^{2} \rangle_{\lambda_{j}} - \frac{1}{2} \sum_{\lambda_{j} \in \Gamma_{\Omega}} \langle \{e_{N}^{2}\}, [\mathbf{c}_{h}] \cdot \mathbf{n} \rangle_{\lambda_{j}} + \frac{1}{2} \langle |\mathbf{c}_{h} \cdot \mathbf{n}|, e_{N}^{2} \rangle_{\partial\Omega_{out}} + \frac{1}{2} \langle |\mathbf{c}_{h} \cdot \mathbf{n}|, e_{N}^{2} \rangle_{\partial\Omega_{in}} = ((\Theta_{N})_{t}, e_{N})_{\Omega} + \left(\nabla \cdot (\mathbf{c}N - \mathbf{c}_{h}\widetilde{N}_{h}), e_{N} \right)_{\Omega} - \sum_{\lambda_{j} \in \Gamma_{\Omega}} \langle [e_{N}(\mathbf{c}N - \mathbf{c}_{h}\widetilde{N}_{h}) \cdot \mathbf{n}_{j}], 1 \rangle_{\lambda_{j}} + \sum_{\lambda_{j} \in \Gamma_{\Omega}} \langle \mathbf{c}N - \widetilde{N}_{h}^{\dagger} \{\mathbf{c}_{h}\} \cdot \mathbf{n}, [e_{N}] \rangle_{\lambda_{j}} + \langle (\mathbf{c}_{h}\widetilde{N}_{h} - \mathbf{c}_{h}\widehat{N}) \cdot \mathbf{n}, e_{N} \rangle_{\partial\Omega_{in}} + \left(\frac{S_{h} - S}{\sigma}, e_{N} \right)_{\Omega}.$$

$$(22)$$

Combine (19), (21) and (22), rearrange terms and use the definition of the L^2 projection to obtain

$$\begin{split} &((e_{\xi})_{t}, e_{\xi})_{\Xi} + ((e_{u})_{t}, e_{u})_{\Xi} + ((e_{N})_{t}, e_{N})_{\Omega} + ||\mu^{1/2} \nabla_{x} e_{u}||_{\Xi}^{2} \\ &+ \frac{1}{2} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle |\{u_{h}\} \cdot n_{i}|, [e_{\xi}]^{2} \rangle_{\gamma_{i}} + \frac{1}{2} \langle |u_{h} \cdot n_{x}|, e_{\xi}^{2} \rangle_{\partial \Xi_{in}} + \frac{1}{2} \langle |u_{h} \cdot n_{x}|, e_{\xi}^{2} \rangle_{\partial \Xi_{out}} \\ &+ \sum_{\gamma_{i} \in \Gamma_{\Xi}} ||\alpha^{1/2} [e_{u}]||_{\gamma_{i}}^{2} + ||\alpha^{1/2} e_{u}||_{\partial \Xi}^{2} + \frac{1}{2} \sum_{\lambda_{j} \in \Gamma_{\Omega}} \langle |\{\mathbf{c}_{h}\} \cdot \mathbf{n}_{j}|, [e_{N}]^{2} \rangle_{\lambda_{j}} \\ &+ \frac{1}{2} \langle |\mathbf{c}_{h} \cdot \mathbf{n}_{j}|, e_{N}^{2} \rangle_{\partial \Omega_{out}} + \frac{1}{2} \langle |\mathbf{c}_{h} \cdot \mathbf{n}|, e_{N}^{2} \rangle_{\partial \Omega_{in}} \\ &= -\frac{1}{2} (\nabla_{x} \cdot u_{h}, e_{\xi}^{2})_{\Xi} + (\nabla_{x} \cdot (uH - u_{h} \widetilde{H}_{h}), e_{\xi})_{\Xi} + (g \nabla_{x} \Theta_{\xi}, e_{u})_{\Xi} + (g e_{\xi}, \nabla_{x} \cdot e_{u})_{\Xi} \\ &+ \mu (\nabla_{x} \Theta_{u}, \nabla_{x} e_{u})_{\Xi} + (u \cdot \nabla_{x} u - u_{h} \cdot \nabla_{x} u_{h}, e_{u})_{\Xi} - \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle g[\Theta_{\xi}], \{e_{u}\} \cdot n_{i} \rangle_{\gamma_{i}} \\ &- \langle g \Theta_{\xi}, e_{u} \cdot n_{x} \rangle_{\partial \Xi_{in}} - \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle g\{e_{\xi}\}, [e_{u}] \cdot n_{i} \rangle_{\gamma_{i}} - \langle ge_{\xi}, e_{u} \cdot n_{x} \rangle_{\partial \Xi_{out}} \\ &+ \frac{1}{2} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle \{e_{\xi}^{2}\}, [u_{h}] \cdot n_{i} \rangle_{\gamma_{i}} - \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle [e_{\xi} (uH - u_{h} \widetilde{H}_{h}) \cdot n_{i}], 1 \rangle_{\gamma_{i}} \end{split}$$

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$$+\sum_{\gamma_{l}\in\Gamma_{\Xi}}\langle(uH-\{u_{h}\}\widetilde{H}_{h}^{\uparrow})\cdot n_{i}, [e_{\xi}]\rangle_{\gamma_{l}}+\langle(u_{h}\widetilde{H}_{h}-u_{h}\widehat{H})\cdot n_{x}, e_{\xi}\rangle_{\partial\Xi_{in}}$$

$$-\sum_{g}\langle|\{u_{h}\}\cdot n_{e}|(u_{h}^{int}-u_{h}^{ext}), e_{u}^{int}\rangle_{\partial\Xi_{g}^{-}}-\sum_{\gamma_{i}\in\Gamma_{\Xi}}\langle\mu\{\nabla_{x}\Theta_{u}\}\cdot n_{i}, [e_{u}]\rangle_{\gamma_{i}}$$

$$+\sum_{\gamma_{i}\in\Gamma_{\Xi}}\langle\mu\{\nabla_{x}e_{u}\}\cdot n_{i}, [\Theta_{u}]\rangle_{\gamma_{i}}+\sum_{\gamma_{i}\in\Gamma_{\Xi}}\langle\alpha[\Theta_{u}], [e_{u}]\rangle_{\gamma_{i}}+\langle\alpha\Theta_{u}, e_{u}\rangle_{\partial\Xi}$$

$$-\mu\langle\nabla_{x}\Theta_{u}\cdot n_{x}, e_{u}\rangle_{\partial\Xi}+\mu\langle\nabla_{x}e_{u}\cdot n_{x}, \Theta_{u}\rangle_{\partial\Xi}+(F_{x,h}-F_{x}, e_{u})_{\Xi}$$

$$+\sum_{\gamma_{i}\in\Gamma_{\Xi}}\langle[S_{xx,h}-S_{xx}], \{e_{u}\}\rangle_{\gamma_{i}}-\frac{1}{2}\left(\nabla\cdot\mathbf{c}_{h}, e_{N}^{2}\right)_{\Omega}+\frac{1}{2}\sum_{\lambda_{j}\in\Gamma_{\Omega}}\langle\{e_{N}^{2}\}, [\mathbf{c}_{h}]\cdot\mathbf{n}_{j}\rangle_{\lambda_{j}}$$

$$+\left(\nabla\cdot(\mathbf{c}N-\mathbf{c}_{h}\widetilde{N}_{h}), e_{N}\right)_{\Omega}-\sum_{\lambda_{j}\in\Gamma_{\Omega}}\langle[e_{N}(\mathbf{c}N-\mathbf{c}_{h}\widetilde{N}_{h})\cdot\mathbf{n}_{j}], 1\rangle_{\lambda_{j}}$$

$$+\sum_{\lambda_{j}\in\Gamma_{\Omega}}\langle\mathbf{c}N-\widetilde{N}_{h}^{\uparrow}\{\mathbf{c}_{h}\}\cdot\mathbf{n}_{j}, [e_{N}]\rangle_{\lambda_{j}}+\langle(\mathbf{c}_{h}\widetilde{N}_{h}-\mathbf{c}_{h}\widehat{N})\cdot\mathbf{n}, e_{N}\rangle_{\partial\Omega_{in}}$$

$$+\left(\frac{S_{h}-S}{\sigma}, e_{N}\right)_{\Omega}.$$
(23)

Integrate the previous equation in time and use the fact that $e_{\xi}(\cdot, 0) = 0$, $e_u(\cdot, 0) = 0$, and $e_N(\cdot, 0) = 0$, and we obtain the error equation

$$\begin{aligned} ||e_{\xi}(T)||_{\Xi}^{2} + ||e_{u}(T)||_{\Xi}^{2} + ||e_{N}(T)||_{\Omega}^{2} + 2\int_{0}^{T} ||\mu^{1/2}\nabla_{x}e_{u}||_{\Xi}^{2}dt \\ + \int_{0}^{T} \sum_{\gamma_{i}\in\Gamma_{\Xi}} \langle |\{u_{h}\}\cdot n_{i}|, [e_{\xi}]^{2}\rangle_{\gamma_{i}}dt + \int_{0}^{T} \langle |u_{h}\cdot n_{x}|, e_{\xi}^{2}\rangle_{\partial\Xi_{in}}dt \\ + \int_{0}^{T} \langle |u_{h}\cdot n_{x}|, e_{\xi}^{2}\rangle_{\partial\Xi_{out}}dt + 2\int_{0}^{T} \sum_{\gamma_{i}\in\Gamma_{\Xi}} ||\alpha^{1/2}[e_{u}]||_{\gamma_{i}}^{2}dt \\ + 2\int_{0}^{T} ||\alpha^{1/2}e_{u}||_{\partial\Xi}^{2}dt + \int_{0}^{T} \sum_{\lambda_{j}\in\Gamma_{\Omega}} \langle |\{\mathbf{c}_{h}\}\cdot\mathbf{n}_{j}|, [e_{N}]^{2}\rangle_{\lambda_{j}}dt \\ + \int_{0}^{T} \langle |\mathbf{c}_{h}\cdot\mathbf{n}|, e_{N}^{2}\rangle_{\partial\Omega_{out}}dt + \int_{0}^{T} \langle |\mathbf{c}_{h}\cdot\mathbf{n}|, e_{N}^{2}\rangle_{\partial\Omega_{in}}dt = 2\sum_{k=1}^{30} E_{k}. \end{aligned}$$

$$(24)$$

5 Error Analysis

We will need the following theorems and identities to perform the error analysis. The following well known theorem [2] will frequently be used:

Theorem 1 Suppose that region R has a Lipschitz boundary. Then, there exists a constant K_R^t such that

$$||v||_{L^{2}(\partial R)} \leq K_{R}^{t} ||v||_{L^{2}(R)}^{1/2} ||v||_{H^{1}(R)}^{1/2} \quad \forall v \in H^{1}(R).$$

$$(25)$$

Let $K_{\Omega}^{t} = \sup_{h} \max_{\Omega_{e} \in \Omega} K_{\Omega_{e}}^{t}$ and $K_{\Xi}^{t} = \sup_{h} \max_{\Xi_{g} \in \Xi} K_{\Xi_{g}}^{t}$, which can be shown to be finite for regular meshes. We define the trace constant $K^{t} = \max(K_{\Omega}^{t}, K_{\Xi}^{t})$. We will need the inverse inequality in which for any functions $v \in V_{h}$ and $w \in W_{h}$

$$||v||_{H^{1}(\mathcal{Z}_{g})} \leq K^{i}_{\mathcal{Z}_{g}} h^{-1}_{g} ||v||_{\mathcal{Z}_{g}},$$
(26)

and

$$||w||_{H^{1}(\Omega_{e})} \leq K_{\Omega_{e}}^{i} h_{e}^{-1} ||w||_{\Omega_{e}},$$
(27)

where K_R^i is independent of h_g , h_e but depends on the shape parameters of region R. Let $K^i = \max_{e,g} (K_{\Omega_e}^i, K_{\Xi_g}^i)$. When the Cauchy-Schwarz inequality is applied to the L^2 product over the domain Ω when one of the functions is only a function of geographic space, we obtain a factor, K^k , of the size of the spectral domain such that

$$|\kappa| \le 2\pi \sigma_{\max} \le K^k. \tag{28}$$

In keeping with the deep water assumption we assume $H^* > H$, H_h , $\tilde{H}_h > H_* > 0$ and that there is a constant K^c such that

$$\begin{aligned} ||h_{b}||_{L^{\infty}(0,T;W_{1}^{\infty}(\Xi))} + ||c_{g}||_{L^{\infty}(0,T;W_{1}^{\infty}(\Omega))} + ||k||_{L^{\infty}(0,T;W_{1}^{\infty}(\Omega))} \\ + \left\| \frac{\sigma}{\sinh 2kH} \right\|_{L^{\infty}(0,T;L^{\infty}(\Omega))} + \left\| \nabla_{\sigma} \left(\frac{\sigma kH}{\sinh 2kH} \right) \right\|_{L^{\infty}(0,T;L^{\infty}(\Omega))} \\ + \left\| \frac{\sigma}{\sinh 2kH_{h}} \right\|_{L^{\infty}(0,T;L^{\infty}(\Omega))} + \left\| \nabla_{\sigma} \left(\frac{\sigma kH_{h}}{\sinh 2kH_{h}} \right) \right\|_{L^{\infty}(0,T;L^{\infty}(\Omega))} \leq K^{c}.$$
(29)

We make the following assumption on the solutions and the projections

$$\begin{aligned} ||H||_{L^{\infty}(0,T;W_{1}^{\infty}(\mathcal{Z}))} + ||u||_{L^{\infty}(0,T;W_{1}^{\infty}(\mathcal{Z}))} + ||N||_{L^{\infty}(0,T;W_{1}^{\infty}(\mathcal{\Omega}))} \\ + ||\tilde{H}_{h}||_{L^{\infty}(0,T;W_{1}^{\infty}(\mathcal{Z}))} + ||\tilde{u}_{h}||_{L^{\infty}(0,T;W_{1}^{\infty}(\mathcal{L}))} + ||\tilde{N}_{h}||_{L^{\infty}(0,T;W_{1}^{\infty}(\mathcal{\Omega}))} \leq K^{m}. \end{aligned}$$
(30)

We also assume that there exists a finite constant $K^M \ge 2K^m$ and independent of h such that

$$||e_{\xi}||_{L^{\infty}(0,T;L^{\infty}(\mathcal{Z}))} + ||e_{u}||_{L^{\infty}(0,T;L^{\infty}(\mathcal{Z}))} + ||e_{N}||_{L^{\infty}(0,T;L^{\infty}(\Omega))} \le K^{M}.$$
 (31)

For the analysis, we will need Young's inequality

$$ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2. \tag{32}$$

We use the basic principles of the following generic arguments throughout the analysis. For functions f, g and ω over region R (which could be either Ω or Ξ), we have that

$$(gf - g_h f_h, \omega)_R = (g(f - f_h) - (g - g_h) f_h, \omega)_R$$

= $(g(\theta_f - e_f) - (\theta_g - e_g) f_h, \omega)_R$
 $\leq C||g||_{L^{\infty}(R)}(||\theta_f||_R + ||e_f||_R + ||\omega||_R)$
+ $C||f_h||_{L^{\infty}(R)}(||\theta_g||_R + ||e_g||_R + ||\omega||_R),$ (33)

and, similarly,

$$(\nabla_{x}(gf - g_{h}f_{h}), \omega)_{R} = (\nabla_{x}(g(f - f_{h}) - (g - g_{h})f_{h}), \omega)_{R}$$

$$= (\nabla_{x}g(\theta_{f} - e_{f}) + g\nabla_{x}(\theta_{f} - e_{f}), \omega)_{R}$$

$$- (\nabla_{x}(\theta_{g} - e_{g})f_{h} - (\theta_{g} - e_{g})\nabla_{x}f_{h}, \omega)_{R}$$

$$\leq ||g||_{L^{\infty}(R)}(||\nabla_{x}\theta_{f}||_{R} + ||\nabla_{x}e_{f}||_{R} + ||\omega||_{R})$$

$$+ C||\nabla_{x}g||_{L^{\infty}(R)}(||\theta_{f}||_{R} + ||e_{f}||_{R} + ||\omega||_{R})$$

$$+ C||f_{h}||_{L^{\infty}(R)}(||\nabla_{x}\theta_{g}||_{R} + ||\nabla_{x}e_{g}||_{R} + ||\omega||_{R})$$

$$+ C||\nabla_{x}f_{h}||_{L^{\infty}(R)}(||\theta_{g}||_{R} + ||e_{g}||_{R} + ||\omega||_{R}).$$
(34)

Before we continue with arguments over boundaries, we introduce some notation. Let \mathcal{E}_i $\{\Xi_g: \partial \Xi_g \cap \gamma_i \neq \emptyset\}$ for each edge γ_i and $\mathscr{E}_i = \{\Omega_e: \partial \Omega_e \cap \lambda_i \neq \emptyset\}$ for each face λ_i . Below, C denotes a positive generic constant, $C(K^*)$ denotes that C depends on K^* , and let ϵ_i , i = 1, 2, 3, 4, be a small generic constant. In the first argument, over the boundary of region R, we use Theorem 1 as well as the inverse and Young's inequalities to obtain

$$\langle f, g \rangle_{\partial R} \leq K^{t} ||f||_{R}^{1/2} ||f||_{H^{1}(R)}^{1/2} ||g||_{R}^{1/2} ||g||_{H^{1}(R)}^{1/2} \leq K^{t} K^{i} h^{-1/2} ||f||_{R}^{1/2} ||f||_{H^{1}(R)}^{1/2} ||g||_{R} \leq C(K^{t}, K^{i}) h^{-1} ||f||_{R} ||f||_{H^{1}(R)} + \epsilon ||g||_{R}^{2} \leq C(K^{t}, K^{i}) (h^{-2} ||f||_{R}^{2} + ||f||_{H^{1}(R)}^{2}) + \epsilon ||g||_{R}^{2},$$
(35)

for $g \in V_h$ or W_h . In the second argument, we utilize the positive constant α . We assume that $\alpha_{i,*}, \alpha_i^*$ are positive parameters such that

$$\alpha_{i,*} \le \alpha|_{\gamma_i} \le \alpha_i^*, \quad \alpha_{i,*}, \alpha_i^* = O\left(h_i^{-1}\right), \tag{36}$$

where

$$h_i = \min_{\Xi_g \subset \mathscr{E}_i} h_g.$$

Then, using α , Theorem 1, and Young's inequality we find

$$\langle f, g \rangle_{\partial R} \leq ||\alpha^{-1/2} f||_{\partial R} ||\alpha^{1/2} g||_{\partial R} \leq K^{t} \alpha_{R,*}^{-1/2} ||f||_{R}^{1/2} ||f||_{H^{1}(R)}^{1/2} ||\alpha^{1/2} g||_{\partial R} \leq C(K^{t}) h_{R} ||f||_{R} ||f||_{H^{1}(R)} + \epsilon ||\alpha^{1/2} g||_{\partial R}^{2} \leq C(K^{t}, K^{i}) ||f||_{R}^{2} + \epsilon ||\alpha^{1/2} g||_{\partial R}^{2},$$
(37)

$$\leq C(K^{t}, K^{t})||f||_{R}^{2} + \epsilon||\alpha^{1/2}g||_{\partial R}^{2},$$
(38)

where one can stop at (37) or apply the inverse inequality if $f \in V_h$ or W_h and obtain (38).

We now continue by estimating the terms on the right hand side of (24), which repeats many arguments from Dawson and Proft [5]. Some of the same arguments are used in the analysis of the action balance equation. By assumptions (30) and (31) we find

$$E_{1} = -\frac{1}{2} \int_{0}^{T} \left(\nabla_{x} \cdot u_{h}, e_{\xi}^{2} \right)_{\Xi} dt$$

$$= -\frac{1}{2} \int_{0}^{T} \left(\nabla_{x} \cdot (e_{u} - \Theta_{u} + u), e_{\xi}^{2} \right)_{\Xi} dt$$

$$\leq \epsilon_{1} \int_{0}^{T} ||\nabla_{x} e_{u}||_{\Xi}^{2} dt + C(K^{M}) \int_{0}^{T} ||\nabla_{x} \Theta_{u}||_{\Xi}^{2} dt + C(K^{m}, K^{M}) \int_{0}^{T} ||e_{\xi}||_{\Xi}^{2} dt.$$
(39)

Again by assumptions (30) and (31) and following the arguments in (34), we have

$$E_{2} = \int_{0}^{T} (\nabla_{x} \cdot (uH - u_{h}\widetilde{H}_{h}), e_{\xi})_{\Xi} dt$$

$$\leq C(K^{m}, K^{M}) \int_{0}^{T} ||\Theta_{u}||_{H^{1}(\Xi)}^{2} dt + C(K^{m}, K^{M}) \int_{0}^{T} ||\Theta_{\xi}||_{H^{1}(\Xi)}^{2} dt$$

$$+\epsilon_{1} \int_{0}^{T} ||\nabla_{x}e_{u}||_{\Xi}^{2} dt + C(K^{m}, K^{M}) \int_{0}^{T} ||e_{\xi}||_{\Xi}^{2} dt.$$
(40)

Using Young's inequality, we find

$$E_{3} + E_{4} + E_{5} = \int_{0}^{T} (g \nabla_{x} \Theta_{\xi}, e_{u})_{\Xi} dt + \int_{0}^{T} (g e_{\xi}, \nabla_{x} \cdot e_{u})_{\Xi} dt + \int_{0}^{T} \mu (\nabla_{x} \Theta_{u}, \nabla_{x} e_{u})_{\Xi} dt \leq C \int_{0}^{T} ||\nabla_{x} \Theta_{\xi}||_{\Xi}^{2} dt + C \int_{0}^{T} ||e_{u}||_{\Xi}^{2} dt + C \int_{0}^{T} ||e_{\xi}||_{\Xi}^{2} dt + \epsilon_{1} \int_{0}^{T} ||\nabla_{x} e_{u}||_{\Xi}^{2} dt + C \int_{0}^{T} ||\nabla_{x} \Theta_{u}||_{\Xi}^{2} dt.$$
(41)

Following the argument in (34), we get

$$E_{6} = \int_{0}^{T} (u \cdot \nabla_{x} u - u_{h} \cdot \nabla_{x} u_{h}, e_{u}) \varepsilon dt$$

$$\leq C(K^{m}) \int_{0}^{T} ||\Theta_{u}||_{\Xi}^{2} dt + C(K^{m}, K^{M}) \int_{0}^{T} (||e_{u}||_{\Xi}^{2} + ||\nabla_{x}\Theta_{u}||_{\Xi}^{2}) dt$$

$$+\epsilon_{1} \int_{0}^{T} ||\nabla_{x} e_{u}||_{\Xi}^{2} dt.$$
(42)

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For E_7 , apply the argument in (35) to obtain

$$E_{7} = \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle g[\Theta_{\xi}], \{e_{u}\} \cdot n_{i} \rangle_{\gamma_{i}} dt$$

$$\leq C(K^{t}, K^{i}) \int_{0}^{T} \sum_{\mathcal{Z}_{g} \subset \Xi} \left[h_{g}^{-2} ||\Theta_{\xi}||_{\mathcal{Z}_{g}}^{2} + ||\Theta_{\xi}||_{H^{1}(\mathcal{Z}_{g})}^{2} \right] dt + C \int_{0}^{T} ||e_{u}||_{\Xi}^{2} dt, \quad (43)$$

with a similar bound for E_8 . Employing the argument in (38) we have

$$E_{9} = -\int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle g\{e_{\xi}\}, [e_{u}] \cdot n_{i} \rangle_{\gamma_{i}} dt$$

$$\leq C \left(K^{t}, K^{i}\right) \int_{0}^{T} ||e_{\xi}||_{\Xi}^{2} dt + \epsilon_{2} \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} ||\alpha^{1/2}[e_{u}]||_{\gamma_{i}}^{2} dt, \qquad (44)$$

and an analogous bound for E_{10} . Following the strategies in (35) and (38) coupled with the fact that [u] = 0 on γ_i , we have

$$\begin{split} E_{11} &= -\frac{1}{2} \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle \{e_{\xi}^{2}\}, [u_{h}] \cdot n_{i} \rangle_{\gamma_{i}} dt \\ &= \frac{1}{2} \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{int}} \langle \{e_{\xi}^{2}\}, ([e_{u}] - [\Theta_{u}]) \cdot n_{i} \rangle_{\gamma_{i}} dt \\ &\leq \epsilon_{2} \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} ||\alpha^{1/2} [e_{u}]||_{\gamma_{i}}^{2} dt + C(K^{t}) \int_{0}^{T} \sum_{S_{g} \subset \Xi} \left(h_{g}^{-2} ||\Theta_{u}||_{\Xi_{g}}^{2} + ||\Theta_{u}||_{H^{1}(\Xi_{g})}^{2}\right) dt \\ &+ C \left(K^{M}, K^{t}, K^{i}\right) \int_{0}^{T} ||e_{\xi}||_{\Xi}^{2} dt. \end{split}$$

$$(45)$$

Use $\{ab\} = \{a\}\{b\} + \frac{1}{4}[a][b]$ and $[ab] = \{a\}[b] + [a]\{b\}$ to obtain

$$\begin{split} E_{12} + E_{13} &= -\int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle [e_{\xi}(uH - u_{h}\widetilde{H}_{h}) \cdot n_{i}], 1 \rangle_{\gamma_{i}} dt \\ &+ \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle (uH - \{u_{h}\}\widetilde{H}_{h}^{\uparrow}) \cdot n_{i}, [e_{\xi}] \rangle_{\gamma_{i}} dt \end{split}$$

$$= \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle \left(\Theta_{\xi}^{\uparrow} - \{\Theta_{\xi}\} \right) \{u_{h}\} \cdot n_{i} - \frac{1}{4} [\Theta_{\xi}][u_{h}] \cdot n_{i}, [e_{\xi}] \rangle_{\gamma_{i}} dt \\ + \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle \{\widetilde{H}_{h}\} ([e_{u}] - [\Theta_{u}]) \cdot n_{i} - [\Theta_{\xi}] \{u_{h}\} \cdot n_{i}, \{e_{\xi}\} \rangle_{\gamma_{i}} dt.$$

We then use (35) to obtain

$$\int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \left\langle \left(\Theta_{\xi}^{\uparrow} - \{\Theta_{\xi}\} \right) \{u_{h}\} \cdot n_{i}, [e_{\xi}] \right\rangle_{\gamma_{i}} dt$$

$$\leq C \left(K^{m}, K^{M}, K^{t} \right) \int_{0}^{T} \sum_{\Xi_{g} \subset \Xi} \left(h_{g}^{-2} ||\Theta_{\xi}||_{\Xi}^{2} + ||\Theta_{\xi}||_{H^{1}(\Xi)}^{2} \right) dt$$

$$+ C \left(K^{m}, K^{M}, K^{t}, K^{i} \right) \int_{0}^{T} ||e_{\xi}||_{\Xi}^{2} dt, \qquad (46)$$

with similar bounds for the two terms with $[\Theta_{\xi}]$ and a nearly identical bound to E_{11} for the remaining term $\int_0^T \sum_{\gamma_i \in \Gamma_{\mathcal{S}}} \langle \{\widetilde{H}_h\} ([e_u] - [\Theta_u]) \cdot n_i, \{e_{\xi}\} \rangle_{\gamma_i} dt$. Applying the Theorem 1 and (32) we have

$$E_{14} = \int_{0}^{T} \langle (u_h \widetilde{H}_h - u_h \widehat{H}) \cdot n_x, e_{\xi} \rangle_{\partial \Xi_{in}} dt$$

$$\leq C \left(K^m, K^M \right) \int_{0}^{T} ||\Theta_{\xi}||_{\partial \Xi_{in}}^2 dt + \epsilon_3 \int_{0}^{T} \langle |u_h \cdot n_x|, e_{\xi}^2 \rangle_{\partial \Xi_{in}} dt$$

$$\leq C \left(K^m, K^M, K^t \right) \int_{0}^{T} \left(||\Theta_{\xi}||_{\Xi}^2 + ||\Theta_{\xi}||_{H^1(\Xi)}^2 \right) dt + \epsilon_3 \int_{0}^{T} \langle |u_h \cdot n_x|, e_{\xi}^2 \rangle_{\partial \Xi_{in}} dt. \quad (47)$$

For E_{15} , use arguments (35) and (38) so that

$$E_{15} = -\int_{0}^{T} \sum_{g} \langle |\{u_{h}\} \cdot n_{g}|(u_{h}^{int} - u_{h}^{ext}), e_{u}^{int} \rangle_{\partial \Xi_{g}^{-}} dt$$

$$\leq \int_{0}^{T} \sum_{g} \langle |\{e_{u}\} + \{\widetilde{u}_{h}\} \rangle \cdot n_{g}||[\Theta_{u}] - [e_{u}]|, |e_{u}^{int}| \rangle_{\partial \Xi_{g}^{-}} dt$$

$$\leq C(K^{m}, K^{M}, K^{t}, K^{i}) \int_{0}^{T} ||e_{u}||_{\Xi}^{2} dt + \epsilon_{2} \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} ||\alpha^{1/2}[e_{u}]||_{\gamma_{i}}^{2} dt$$

$$+ C(K^{t}) \int_{0}^{T} \sum_{\Xi_{g} \in \Xi} (h_{g}^{-2}||\Theta_{u}||_{\Xi}^{2} + ||\Theta_{u}||_{H^{1}(\Xi)}^{2}) dt + \epsilon_{4} \int_{0}^{T} ||\alpha^{1/2}e_{u}||_{\partial\Xi}^{2} dt, \quad (48)$$

and use (37) to find

$$E_{16} = -\int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle \mu \{ \nabla_{x} \Theta_{u} \} \cdot n_{i}, [e_{u}] \rangle_{\gamma_{i}} dt$$

$$\leq C(K^{t}) \int_{0}^{T} \sum_{\mathcal{S}_{g} \subset \Xi} h_{g} || \nabla_{x} \Theta_{u} ||_{\mathcal{S}_{g}} || \nabla_{x} \Theta_{u} ||_{H^{1}(\mathcal{S}_{g})} dt + \epsilon_{2} \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} || \alpha^{1/2} [e_{u}] ||_{\gamma_{i}}^{2} dt.$$

$$(49)$$

Similarly, (35) gives us that

$$E_{17} = \int_{0}^{T} \sum_{\gamma_i \in \Gamma_{\Xi}} \langle \mu \{ \nabla_x e_u \} \cdot n_i, [\Theta_u] \rangle_{\gamma_i} dt$$

$$\leq \epsilon_1 \int_{0}^{T} ||\nabla_x e_u||_{\mathcal{Z}_g}^2 dt + C(K^t, K^i) \int_{0}^{T} \sum_{\mathcal{Z}_g \subset \Xi} (h_g^{-2} ||\Theta_u||_{\mathcal{Z}_g}^2 + ||\Theta_u||_{H^1(\mathcal{Z}_g)}^2) dt.$$
(50)

Simply using Theorem 1, (26), and (36) we get

$$E_{18} + E_{19} = \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle \alpha[\Theta_{u}], [e_{u}] \rangle_{\gamma_{i}} dt + \int_{0}^{T} \langle \alpha \Theta_{u}, e_{u} \rangle_{\partial \Xi} dt$$

$$\leq C \int_{0}^{T} \Big(\sum_{\gamma_{i} \in \Gamma_{\Xi}} ||\alpha^{1/2}[\Theta_{u}]||_{\gamma_{i}}^{2} + ||\alpha^{1/2}[\Theta_{u}]||_{\partial \Xi}^{2} \Big) dt$$

$$+ \epsilon_{2} \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} ||\alpha^{1/2}[e_{u}]||_{\gamma_{i}}^{2} dt + \epsilon_{4} \int_{0}^{T} ||\alpha^{1/2}e_{u}||_{\partial \Xi}^{2} dt$$

$$\leq C(K^{t}) \int_{0}^{T} \sum_{\Xi_{g} \subset \Xi} (h_{g}^{-2}||\Theta_{u}||_{\Xi_{g}}^{2} + ||\Theta_{u}||_{H^{1}(\Xi_{g})}^{2}) dt$$

$$+ \epsilon_{2} \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} ||\alpha^{1/2}[e_{u}]||_{\gamma_{i}}^{2} dt + \epsilon_{4} \int_{0}^{T} ||\alpha^{1/2}e_{u}||_{\partial \Xi}^{2} dt, \qquad (51)$$

and using (35) and (38) we have

$$E_{20} + E_{21} = -\int_{0}^{T} \langle \mu \nabla_{x} \Theta_{u} \cdot n, e_{u} \rangle_{\partial \Xi} dt + \int_{0}^{T} \langle \mu \nabla_{x} e_{u} \cdot n, \Theta_{u} \rangle_{\partial \Xi} dt$$
$$\leq \epsilon_{1} \int_{0}^{T} ||\nabla_{x} e_{u}||_{\Xi_{g}}^{2} dt + \epsilon_{4} \int_{0}^{T} ||\alpha^{1/2} e_{u}||_{\partial \Xi}^{2} dt$$

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$$+C(K^{t})\int_{0}^{T}\sum_{\mathcal{Z}_{g}\subset\mathcal{Z}}h_{g}||\nabla_{x}\Theta_{u}||_{\mathcal{Z}_{g}}||\nabla_{x}\Theta_{u}||_{H^{1}(\mathcal{Z}_{g})}dt$$
$$+C(K^{t},K^{i})\int_{0}^{T}\sum_{\mathcal{Z}_{g}\subset\mathcal{Z}}(h_{g}^{-2}||\Theta_{u}||_{\mathcal{Z}_{g}}^{2}+||\Theta_{u}||_{H^{1}(\mathcal{Z}_{g})}^{2})dt.$$
(52)

Integrating E_{22} by parts, using the definition of F_x , and applying the argument in (38) on the boundary terms we obtain

$$E_{22} + E_{23} = \int_{0}^{T} \left(F_{x,h} - F_{x}, e_{u} \right)_{\Xi} dt + \sum_{\gamma_{i} \in \Gamma_{\Xi}} \left\langle [S_{xx,h} - S_{xx}], \{e_{u}\} \cdot n_{i} \right\rangle_{\gamma_{i}}$$

$$= \int_{0}^{T} \left(\int_{\theta_{\min}}^{\theta_{\max}} \int_{\sigma_{\min}}^{\sigma_{\max}} \left(\frac{1}{2} \cos^{2} \theta \sigma (N_{h} - N) \right) d\sigma d\theta, \nabla_{x} \left(\frac{g}{h_{b}} e_{u} \right) \right)_{\Xi} dt$$

$$- \sum_{\gamma_{i} \in \Gamma_{\Xi}} \int_{0}^{T} \left\langle \int_{\theta_{\min}}^{\theta_{\max}} \int_{\sigma_{\min}}^{\sigma_{\max}} \left(\frac{1}{2} \cos^{2} \theta \sigma \{N_{h} - N\} \right) d\sigma d\theta, \frac{g}{h_{b}} [e_{u}] \cdot n_{i} \right\rangle_{\gamma_{i}} dt$$

$$- \int_{0}^{T} \left\langle \int_{\theta_{\min}}^{\theta_{\max}} \int_{\sigma_{\min}}^{\sigma_{\max}} \left(\frac{1}{2} \cos^{2} \theta \sigma (N_{h} - N) \right) d\sigma d\theta, \frac{g}{h_{b}} e_{u} \cdot n_{x} \right\rangle_{\partial \Xi} dt$$

$$\leq C(K^{c}) \int_{0}^{T} ||e_{u}||_{\Xi}^{2} dt + \epsilon_{1} \int_{0}^{T} ||\nabla_{x} e_{u}||_{\Xi}^{2} dt$$

$$+ C(K^{t}, K^{i}, K^{c}, K^{k}) \int_{0}^{T} (||e_{N}||_{\Omega}^{2} + ||\Theta_{N}||_{\Omega}^{2}) dt$$

$$+ \epsilon_{2} \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} ||\alpha^{1/2} [e_{u}]||_{\gamma_{i}} dt + \epsilon_{4} \int_{0}^{T} ||\alpha^{1/2} e_{u}||_{\partial \Xi} dt.$$
(53)

Expanding terms and defining $B(H) = \nabla_{\sigma} \frac{k\sigma H}{\sinh(2kH)}$ which is a Lipschitz continuous function, we have

$$E_{24} = -\frac{1}{2} \int_{0}^{T} \left(\nabla \cdot \mathbf{c}_{h}, e_{N}^{2} \right)_{\Omega} dt$$

$$= -\frac{1}{2} \int_{0}^{T} \left(\cos \theta \nabla_{x} c_{g}, e_{N}^{2} \right)_{\Omega} dt - \frac{1}{2} \int_{0}^{T} \left((1 + \frac{1}{2} \cos^{2} \theta + \cos 2\theta) \nabla_{x} u_{h}, e_{N}^{2} \right)_{\Omega} dt$$

$$+ \frac{1}{2} \int_{0}^{T} \left(\nabla_{x} u_{h} B(H_{h}), e_{N}^{2} \right)_{\Omega} dt - \frac{1}{2} \int_{0}^{T} \left(\frac{\sigma}{\sinh(2kH_{h})} \nabla_{x} h_{b} \cos \theta, e_{N}^{2} \right)_{\Omega} dt$$

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$$= -\frac{1}{2} \int_{0}^{T} (\cos \theta \nabla_{x} c_{g}, e_{N}^{2})_{\Omega} dt$$

$$-\frac{1}{2} \int_{0}^{T} \left((1 + \frac{1}{2} \cos^{2} \theta + \cos 2\theta) \nabla_{x} (e_{u} - \Theta_{u} + u), e_{N}^{2} \right)_{\Omega} dt$$

$$+\frac{1}{2} \int_{0}^{T} \left(\nabla_{x} (e_{u} - \Theta_{u} + u) B(e_{\xi} - \Theta_{\xi} + H), e_{N}^{2} \right)_{\Omega} dt$$

$$-\frac{1}{2} \int_{0}^{T} \left(\frac{\sigma}{\sinh(2kH_{h})} \nabla_{x} h_{b} \cos \theta, e_{N}^{2} \right)_{\Omega} dt$$

$$\leq C(K^{M}, K^{k}) \int_{0}^{T} ||\nabla_{x} \Theta_{u}||_{\Xi}^{2} dt + C(K^{m}, K^{M}, K^{c}, K^{k}) \int_{0}^{T} ||e_{N}||_{\Omega}^{2} dt$$

$$+\epsilon_{1} \int_{0}^{T} ||\nabla_{x} e_{u}||_{\Xi}^{2} dt + C(K^{m}, K^{M}, K^{k}) \int_{0}^{T} (||e_{\xi}||_{\Xi}^{2} + ||\Theta_{\xi}||_{\Xi}^{2} + ||\nabla_{x} \Theta_{u}||_{\Xi}^{2}) dt.$$
(54)

Since $[(c_g \cos \theta, c_{\sigma,h}, c_{\theta,h})] = 0$ on λ_j , we follow E_{11} to get

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$$E_{25} = \frac{1}{2} \sum_{\lambda_j \in \Gamma_{\Omega}} \langle \{e_N^2\}, [\mathbf{c}_h] \cdot \mathbf{n} \rangle_{\lambda_j} = \frac{1}{2} \sum_{\gamma_l \in \Gamma_{\Sigma}} \langle \{e_N^2\}, [u_h] \cdot n_i \rangle_{\gamma_l \times \kappa}$$

$$\leq \epsilon_2 \int_0^T \sum_{\gamma_l \in \Gamma_{\Sigma}} ||\alpha^{1/2} [e_u]||_{\gamma_l}^2 dt + C(K^t) \int_0^T \sum_{\mathcal{E}_g \subset \mathcal{E}} \left[h_g^{-2} ||\Theta_u||_{\mathcal{E}_g}^2 + ||\Theta_u||_{H^1(\mathcal{E}_g)}^2 \right] dt$$

$$+ C(K^M, K^t, K^i, K^k) \int_0^T ||e_N||_{\Omega}^2 dt.$$
(55)

Define $g(H) = -\frac{k\sigma}{\sinh(2kH)}H - \cos^2\theta c_g k$, which is a Lipschitz continuous function, and expand to obtain

$$E_{26} = \left(\nabla \cdot (\mathbf{c}N - \mathbf{c}_{h}\widetilde{N}_{h}), e_{N}\right)_{\Omega}$$

$$= \int_{0}^{T} \left(\cos\theta c_{g}\nabla_{x}(N - \widetilde{N}_{h}), e_{N}\right)_{\Omega} dt + \int_{0}^{T} \left(\nabla_{x}(uN - u_{h}\widetilde{N}_{h}), e_{N}\right)_{\Omega} dt$$

$$+ \int_{0}^{T} \left(\nabla_{\sigma} \left(g(H)\nabla_{x}uN - g(H_{h})\nabla_{x}u_{h}\widetilde{N}_{h}\right), e_{N}\right)_{\Omega} dt$$

$$+ \int_{0}^{T} \left(\nabla_{\theta} \left(\frac{\sigma}{\sinh(2kH)}\nabla_{x}h_{b}\sin\theta N - \frac{\sigma}{\sinh(2kH_{h})}\nabla_{x}h_{b}\sin\theta\widetilde{N}_{h}\right), e_{N}\right)_{\Omega} dt$$

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$$+ \int_{0}^{T} \left(\nabla_{\theta} (\nabla_{x} u \cos \theta \sin \theta N - \nabla_{x} u_{h} \cos \theta \sin \theta \widetilde{N}_{h}), e_{N} \right)_{\Omega} dt$$
$$= \sum_{i=1}^{5} E_{26,i}, \tag{56}$$

where

$$E_{26,1} = \int_{0}^{T} \left(\cos\theta c_g \nabla_x \Theta_N, e_N\right)_{\Omega} dt \le C(K^c) \int_{0}^{T} \left(||\nabla_x \Theta_N||_{\Omega}^2 + ||e_N||_{\Omega}^2\right) dt.$$
(57)

Following E_2 , we use (34) and have

$$E_{26,2} = \int_{0}^{T} (\nabla_{x} \cdot (uN - u_{h}\widetilde{N}_{h}), e_{N})_{\Omega} dt$$

$$\leq C(K^{m}, K^{M}, K^{k}) \int_{0}^{T} (||\Theta_{u}||^{2}_{H^{1}(\Sigma)} + ||\Theta_{N}||^{2}_{H^{1}(\Omega)} + ||e_{N}||^{2}_{\Omega}) dt$$

$$+\epsilon_{1} \int_{0}^{T} ||\nabla_{x}e_{u}||^{2}_{\Xi} dt.$$
(58)

For the following term, we first note that since the group velocity, c_g , and wave number, k, depend on the relative frequency, σ , that g(H) also depends on the relative frequency σ as well as the total water depth H. Then following a similar argument in (34) with more terms, we expand terms and use assumptions (28)–(31) to obtain

$$\begin{split} E_{26,3} &= \int_{0}^{T} \left(\nabla_{\sigma} \left(g(H) \nabla_{x} u N - g(H_{h}) \nabla_{x} u_{h} \widetilde{N}_{h} \right), e_{N} \right)_{\Omega} dt \\ &= \int_{0}^{T} \left(\nabla_{x} u N \nabla_{\sigma} (g(H) - g(H_{h})) + \nabla_{x} u (g(H) - g(H_{h})) \nabla_{\sigma} N, e_{N} \right)_{\Omega} dt \\ &+ \int_{0}^{T} \left(\nabla_{x} (\Theta_{u} - e_{u}) \widetilde{N}_{h} \nabla_{\sigma} g(H_{h}) + \nabla_{x} (\Theta_{u} - e_{u}) g(H_{h}) \nabla_{\sigma} \widetilde{N}_{h}, e_{N} \right)_{\Omega} dt \\ &+ \int_{0}^{T} \left(\nabla_{x} u \nabla_{\sigma} (g(H)) \Theta_{N} + \nabla_{x} u g(H) \nabla_{\sigma} \Theta_{N}, e_{N} \right)_{\Omega} dt \\ &\leq C(K^{m}, K^{k}) \int_{0}^{T} \left(||e_{\xi}||_{\Xi}^{2} + ||\Theta_{\xi}||_{\Xi}^{2} \right) dt + \epsilon_{1} \int_{0}^{T} ||\nabla_{x} e_{u}||_{\Omega}^{2} dt \end{split}$$

$$+C(K^{m}, K^{M}, K^{c}) \int_{0}^{T} ||\nabla_{x} \Theta_{u}||_{\mathcal{Z}}^{2} dt + C(K^{m}, K^{c}, K^{k}) \int_{0}^{T} ||\Theta_{N}||_{\Omega}^{2} dt +C\left(K^{m}, K^{M}, K^{c}, K^{k}\right) \int_{0}^{T} \left(||\nabla_{\sigma} \Theta_{N}||_{\Omega}^{2} + ||e_{N}||_{\Omega}^{2} \right) dt.$$
(59)

Again, expanding terms, using assumptions (28)-(31) and the argument in (34), we find

$$E_{26,4} = \int_{0}^{T} \left(\nabla_{\theta} \left(\frac{\sigma}{\sinh(2kH)} \nabla_{x} h_{b} \sin\theta N - \frac{\sigma}{\sinh(2kH_{h})} \nabla_{x} h_{b} \sin\theta \widetilde{N}_{h} \right), e_{N} \right)_{\Omega} dt$$

$$\leq C \left(K^{c}, K^{k} \right) \int_{0}^{T} ||\Theta_{N}||_{\Omega}^{2} dt + C \left(K^{m}, K^{M}, K^{c}, K^{k} \right) \int_{0}^{T} (||e_{\xi}||_{\Xi}^{2} + ||\Theta_{\xi}||_{\Xi}^{2}) dt$$

$$+ C (K^{m}, K^{c}, K^{k}) \int_{0}^{T} ||e_{N}||_{\Omega}^{2} dt + C (K^{M}, K^{c}, K^{k}) \int_{0}^{T} ||\nabla_{\theta}\Theta_{N}||_{\Omega}^{2} dt, \qquad (60)$$

and

$$E_{26,5} = \int_{0}^{T} \left(\nabla_{\theta} (\nabla_{x} u \cos \theta \sin \theta N - \nabla_{x} u_{h} \cos \theta \sin \theta \widetilde{N}_{h}), e_{N} \right)_{\Omega} dt$$

$$\leq C(K^{m}) \int_{0}^{T} \left(||\Theta_{N}||_{\Omega}^{2} + ||e_{N}||_{\Omega}^{2} \right) dt + \epsilon_{1} \int_{0}^{T} ||\nabla_{x} e_{u}||_{\Xi}^{2}$$

$$+ C\left(K^{m}, K^{M}\right) \int_{0}^{T} \left(||\nabla_{x} \Theta_{u}||_{\Xi}^{2} + ||\nabla_{\theta} \Theta_{N}||_{\Omega}^{2} \right) dt.$$
(61)

Like E_{12} and E_{13} , we use $\{ab\} = \{a\}\{b\} + \frac{1}{4}[a][b]$ and $[ab] = \{a\}[b] + [a]\{b\}$ to obtain

$$E_{27} + E_{28} = -\sum_{\lambda_j \in \Gamma_{\Omega}} \langle [e_N(\mathbf{c}N - \mathbf{c}_h \widetilde{N}_h) \cdot \mathbf{n}], 1 \rangle_{\lambda_j} + \sum_{\lambda_j \in \Gamma_{\Omega}} \langle \mathbf{c}N - \widetilde{N}_h^{\uparrow} \{\mathbf{c}_h\} \cdot \mathbf{n}, [e_N] \rangle_{\lambda_j}$$
$$= \sum_{\lambda_j \in \Gamma_{\Omega}} \langle (\Theta_N^{\uparrow} - \{\Theta_N\}) \{\mathbf{c}_h\} \cdot \mathbf{n} + \frac{1}{4} [\Theta_N] [\mathbf{c}_h] \cdot \mathbf{n}, [e_N] \rangle_{\lambda_j}$$
$$+ \sum_{\lambda_j \in \Gamma_{\Omega}} \langle \{\widetilde{N}_h\} [\mathbf{c}_h] \cdot \mathbf{n} + [\Theta_N] \{\mathbf{c}_h\} \cdot \mathbf{n}, \{e_N\} \rangle_{\lambda_j}.$$
(62)

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Writing $\{\mathbf{c}_h\} = \{\mathbf{c}_h - \mathbf{c}\} + \{\mathbf{c}\}$ and following (35), we find

$$\sum_{\lambda_{j}\in\Gamma_{\Omega}}\langle(\Theta_{N}^{\uparrow}-\{\Theta_{N}\})\{\mathbf{c}_{h}\}\cdot\mathbf{n},[e_{N}]\rangle_{\lambda_{j}}$$

$$\leq C(K^{m},K^{M},K^{t},K^{c})\int_{0}^{T}\sum_{\Omega_{e}\subset\Omega}\left(h_{e}^{-2}||\Theta_{N}||_{\Omega}^{2}+||\Theta_{N}||_{H^{1}(\Omega)}^{2}\right)dt$$

$$+\epsilon_{1}\int_{0}^{T}||\nabla_{x}e_{u}||_{\Xi}^{2}dt+C\left(K^{M},K^{t},K^{i},K^{c}\right)\int_{0}^{T}||\nabla_{x}\Theta_{u}||_{\Xi}^{2}dt$$

$$+C\left(K^{m},K^{M},K^{t},K^{i},K^{c}\right)\int_{0}^{T}||e_{N}||_{\Omega}^{2}dt,$$
(63)

with a similar bound for the other term in (62) with $\{c_h\}$. As with E_{25} , we have that

$$\sum_{\lambda_{j}\in\Gamma_{\Omega}} \left\langle \frac{1}{4} [\Theta_{N}][\mathbf{c}_{h}] \cdot \mathbf{n}, [e_{N}] \right\rangle_{\lambda_{j}} + \sum_{\lambda_{j}\in\Gamma_{\Omega}} \left\langle \{\widetilde{N}_{h}\}[\mathbf{c}_{h}] \cdot \mathbf{n}, \{e_{N}\} \right\rangle_{\lambda_{j}}$$

$$= \sum_{\gamma_{i}\in\Gamma_{\Sigma}} \left\langle \frac{1}{4} [\Theta_{N}][u_{h}-u] \cdot n_{i}, [e_{N}] \right\rangle_{\gamma_{i}\times\kappa} + \sum_{\gamma_{i}\in\Gamma_{\Sigma}} \left\langle \{\widetilde{N}_{h}\}[u_{h}-u] \cdot n_{i}, \{e_{N}\} \right\rangle_{\gamma_{i}\times\kappa}$$

$$\leq \epsilon_{2} \int_{0}^{T} \sum_{\gamma_{i}\in\Gamma_{\Sigma}} ||\alpha^{1/2}[e_{u}]||_{\gamma_{i}}^{2} dt + C(K^{t}) \int_{0}^{T} \sum_{\Xi_{g}\subset\Sigma} [h_{g}^{-2}||\Theta_{u}||_{\Xi_{g}}^{2} + ||\Theta_{u}||_{H^{1}(\Xi_{g})}^{2}] dt$$

$$+ C(K^{m}, K^{M}, K^{t}, K^{i}, K^{k}) \int_{0}^{T} ||\Theta_{N}||_{\Omega}^{2} dt.$$
(64)

In the same way,

$$E_{29} = \langle (\mathbf{c}_{h}\widetilde{N}_{h} - \mathbf{c}_{h}\widehat{N}) \cdot \mathbf{n}, e_{N} \rangle_{\partial \Omega_{in}}$$

$$\leq C\left(K^{m}, K^{M}, K^{t}, K^{c}\right) \int_{0}^{T} \sum_{\Omega_{e} \subset \Omega} \left(h_{e}^{-2} ||\Theta_{N}||_{\Omega}^{2} + ||\Theta_{N}||_{H^{1}(\Omega)}^{2}\right) dt$$

$$+\epsilon_{1} \int_{0}^{T} ||\nabla_{x}e_{u}||_{\Xi}^{2} dt + C\left(K^{M}, K^{t}, K^{i}, K^{c}\right) \int_{0}^{T} ||\nabla_{x}\Theta_{u}||_{\Xi}^{2} dt$$

$$+C\left(K^{m}, K^{M}, K^{t}, K^{i}, K^{c}\right) \int_{0}^{T} ||e_{N}||_{\Omega}^{2} dt.$$
(65)

For E_{30} , we use the definition of S and apply the argument in (33) for both f_S and g_S to find

$$E_{30} = \left(\frac{S_h - S}{\sigma}, e_N\right)_{\Omega} = (f_S(H_h)N_h - f_S(H)N + g_S(N_h)N_h - g_S(N)N, e_N)_{\Omega}$$

$$\leq C(K^m, K^M, K^k) \int_0^T (||e_{\xi}||_{\Xi}^2 + ||\Theta_{\xi}||_{\Omega}^2) dt$$

$$+ C(K^m, K^M) \int_0^T (||\Theta_N||_{\Omega}^2 + ||e_N||_{\Omega}^2) dt.$$
(66)

Combining (39)–(66) with (24), choosing ϵ_i , i = 1, 4 sufficiently small, we obtain

$$\begin{split} ||e_{\xi}(T)||_{\Xi}^{2} + ||e_{u}(T)||_{\Xi}^{2} + ||e_{N}(T)||_{\Omega}^{2} + 2\int_{0}^{T} ||\mu^{1/2} \nabla_{x} e_{u}||_{\Xi}^{2} dt \\ + \int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} \langle |\{u_{h}\} \cdot n_{i}|, [e_{\xi}]^{2} \rangle_{\gamma_{i}} dt + \int_{0}^{T} \langle |u_{h} \cdot n_{x}|, e_{\xi}^{2} \rangle_{\partial \Xi_{in}} dt \\ + \int_{0}^{T} \langle |u_{h} \cdot n_{x}|, e_{\xi}^{2} \rangle_{\partial \Xi_{out}} dt + 2\int_{0}^{T} \sum_{\gamma_{i} \in \Gamma_{\Xi}} ||\alpha^{1/2} [e_{u}]||_{\gamma_{i}}^{2} dt \\ + 2\int_{0}^{T} ||\alpha^{1/2} e_{u}||_{\partial \Xi}^{2} dt + \int_{0}^{T} \sum_{\lambda_{j} \in \Gamma_{\Omega}} \langle |\{c_{h}\} \cdot \mathbf{n}_{j}|, [e_{N}]^{2} \rangle_{\lambda_{j}} dt \\ + \int_{0}^{T} \langle |\mathbf{c}_{h} \cdot \mathbf{n}|, e_{N}^{2} \rangle_{\partial \Omega_{out}} dt + \int_{0}^{T} \langle |\mathbf{c}_{h} \cdot \mathbf{n}|, e_{N}^{2} \rangle_{\partial \Omega_{in}} dt \\ \leq C\int_{0}^{T} ||e_{\xi}||_{\Xi}^{2} dt + C\int_{0}^{T} ||e_{u}||_{\Xi}^{2} dt + C\int_{0}^{T} ||e_{N}||_{\Omega}^{2} dt \\ + C^{*} \int_{0}^{T} \sum_{\Xi_{g} \subset \Xi} \langle h_{g}^{-2}||\Theta_{g}||_{\Xi_{g}}^{2} + ||\Theta_{g}||_{H^{1}(\Xi_{g})}^{2}) dt \\ + C^{*} \int_{0}^{T} \sum_{\Xi_{g} \subset \Xi} \langle h_{g}^{-2}||\Theta_{N}||_{\Xi_{g}}^{2} + ||\Theta_{N}||_{H^{1}(\Xi_{g})}^{2}) dt \\ + C^{*} \int_{0}^{T} \sum_{\Xi_{g} \subset \Xi} h_{g}||\nabla_{x} \Theta_{u}||_{\Xi_{g}}||\nabla_{x} \Theta_{u}||_{H^{1}(\Xi_{g})} dt \end{split}$$

$$(67)$$

where $C^* = C(K^m, K^M, K^t, K^i, K^c, K^k)$. Using standard approximation theory results, we find

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$$\int_{0}^{T} \sum_{\mathcal{E}_{g}\subset\mathcal{Z}} \left(h_{g}^{-2} ||\Theta_{\xi}||_{\mathcal{E}_{g}}^{2} + ||\Theta_{\xi}||_{H^{1}(\mathcal{E}_{g})}^{2}\right) dt + \int_{0}^{T} \sum_{\mathcal{E}_{g}\subset\mathcal{Z}} \left(h_{g}^{-2} ||\Theta_{u}||_{\mathcal{E}_{g}}^{2} + ||\Theta_{u}||_{H^{1}(\mathcal{E}_{g})}^{2}\right) dt \\
+ \int_{0}^{T} \sum_{\mathcal{E}_{g}\subset\mathcal{Z}} h_{g} ||\nabla_{x}\Theta_{u}||_{\mathcal{E}_{g}} ||\nabla_{x}\Theta_{u}||_{H^{1}(\mathcal{E}_{g})} dt \\
\leq Ch^{2p} \int_{0}^{T} \left(||\xi||_{H^{p+1}(\mathcal{E})}^{2} + ||u||_{H^{p+1}(\mathcal{E})}^{2}\right) dt = C(K^{r})h^{2p},$$
(68)

and

$$\int_{0}^{T} \sum_{\Omega_{e} \subset \Omega} \left(h_{e}^{-2} ||\Theta_{N}||_{\Omega_{e}}^{2} + ||\Theta_{N}||_{H^{1}(\Omega_{e})}^{2} \right) dt$$

$$\leq Ch^{2\min\{p,q\}} \int_{0}^{T} ||N||_{H^{\min\{p,q\}+1}(\Omega)}^{2} dt = C(K^{r})h^{2\min\{p,q\}}.$$
(69)

We assume the initial data is sufficiently smooth so that

$$\begin{aligned} ||\Theta_{\xi}(\cdot,0)||^{2}_{H^{1}(\varSigma)} + ||\Theta_{u}(\cdot,0)||^{2}_{H^{1}(\varXi)} + ||\Theta_{N}(\cdot,0)||^{2}_{H^{1}(\Omega)} \\ &\leq Ch^{\min\{p,q\}} \left(||\xi_{0}||^{2}_{H^{p+1}(\varXi)} + ||u_{0}||^{2}_{H^{p+1}(\varXi)} \right) + ||N_{0}||^{2}_{H^{\min\{p,q\}+1}(\Omega)} \\ &\leq C(K^{r,0})h^{2\min\{p,q\}}. \end{aligned}$$

$$(70)$$

Now applying Gronwall's inequality to (67) followed by the triangle inequality we obtain the following a priori error estimate:

Theorem 2 Assume, ξ , u, N, and initial data are sufficiently smooth so (68) to (70) hold. Then there exists a constant

$$\hat{C} = C\left(K^{m}, K^{M}, K^{t}, K^{i}, K^{c}, K^{k}, K^{r}, K^{r,0}, T\right),$$

such that

$$\begin{aligned} ||\xi - \xi_h||_{L^{\infty}(0,T;L^2(\mathcal{S}))} + ||u - u_h||_{L^{\infty}(0,T;L^2(\mathcal{S}))} + ||u - u_h||_{L^{\infty}(0,T;H^1(\mathcal{S}))} \\ + ||N - N_h||_{L^{\infty}(0,T;L^2(\Omega))} \le \hat{C}h^{\min\{p,q\}}. \end{aligned}$$
(71)

Lastly, we show that we can remove the dependence of \hat{C} on K^M . To do this we first recall another inverse inequality

$$||v(\cdot, t)||_{L^{\infty}(R)} \leq Kh^{-d/2}||v(\cdot, t)||_{R},$$

where d is the dimension of region R. Assuming p, q > 2 and h sufficiently small, we have

.

$$||e_{\xi}||_{L^{\infty}(0,T;L^{\infty}(\mathcal{Z}))} \leq K\hat{C}h^{p-1} \ll K^{M},$$

with a similar bound for e_u and

$$||e_N||_{L^{\infty}(0,T;L^{\infty}(\Omega))} \leq K\hat{C}h^{\min(p,q)-3/2} \ll K^M,$$

and therefore we can remove the dependence of \hat{C} on K^M [11]. We simplified the analysis by only considering one geographic dimension which in turn simplified the propagation velocities, **c**; however, all of the previous arguments stand for the two-geographic dimension problem.

6 Numerical Results

In this section, we investigate the experimental convergence rates for the DG method applied to the wave model, using a manufactured solution. We also verify the model with several analytic test cases which can be found in the ONR test bed [24]. These analytic test cases are physically one-dimensional but computed in the two-dimensional geographic space. We continue by showing preliminary numerical results of the DG spectral wave model loosely coupled to a DG circulation model (which is described in [4]). All numerical results are computed in the full two-dimensional geographic space, even though the analytic test cases for the spectral wave model are physically only one-dimensional. For the wave model, geographic space is discretized with unstructured triangular elements and spectral space is discretized with structured triangular elements.

6.1 Wave Model

6.1.1 Manufactured Solutions

We first examine individually the wave model with manufactured solutions. To inspect the convergence rates, β , we simplify the action balance equation by setting the propagation velocity to $\mathbf{c} = \mathbf{1}$ and the source term to S = 0 and use the manufactured solution

$$N = \sin(x - t) + \cos(y - t) + \sin(\sigma - t) + \cos(\theta - t),$$

where x, y, σ , θ and t are dimensionless. The geographic and spectral domains are (0, 10) × (0, 10) and are discretized with triangles and quadrilaterals, respectively, with element size h. The series of geographic meshes used all have the same structure. The first geographic mesh, with h = L/2 is shown in Fig. 2. A second order Runge–Kutta scheme is used with a time step of $\Delta t = 0.005$ s, so that errors due to the time discretization are negligible. The errors and convergence rates are shown in Table 1 for the initial condition. For T = 5s, the

Fig. 2 The first mesh of a sequence of geographic meshes, all with the same structure, used for the manufactured solutions. The mesh shown here has an element size of h = L/2 sequence of geographic meshes for the manufactured solutions



h	p, q=0		p, q=1		p, q=2		p, q=3		p, q=4	
	L2 error	β	L2 error	β	L2 error	β	L2 error	β	L2 error	β
L/2	146.6797	_	87.7164	-	40.1412	-	12.5245	-	2.0678	-
L/4	88.4044	0.7305	27.4349	1.6768	5.9322	2.7584	0.9200	3.7670	7.2995e-2	4.8242
L/8	46.0436	0.9411	7.2291	1.9241	0.7747	2.9369	5.9748e-2	3.9447	2.3583e-3	4.9520
L/16	23.2591	0.9852	1.8311	1.9811	9.7896e-2	2.9843	3.7700e-3	3.9863	7.4134e-5	4.9915
L/32	11.6595	0.9963	0.4593	1.9952	1.2270e-2	2.9961	2.3619e-4	3.9965	2.2390e-6	5.0492

 Table 1
 Error in the initial condition for the manufactured solution

Table 2 Error and convergence rates for the manufactured solution at T=5s

h	p, q=0		p, q=1		p, q=2		p, q=3		p, q=4	
	L2 error	β	L2 error	β	L2 error	β	L2 error	β	L2 error	β
L/2	134.5962	_	111.0265	_	53.8817	_	18.1940	_	4.0976	_
L/4	115.8146	0.2168	45.6310	1.2828	8.8633	2.6039	1.3717	3.7294	0.1453	4.8177
L/8	86.3410	0.4237	12.3717	1.8830	1.1659	2.9264	8.9412e-2	3.9394	4.9555e-3	4.8739
L/16	57.8860	0.5768	3.0650	2.0131	0.1479	2.9788	5.9012e-3	3.9214	-	-
L/32	35.2341	0.7162	0.7643	2.0037	1.8667e-2	2.9861	-	-	-	-

error and convergence rates are recorded in Table 2 and the errors are plotted against the step size h and number of degrees of freedom in Fig. 3. For h small enough, we obtain about p + 1 convergence rates.

Restricting to the geographic domain, we use the manufactured solution

$$N = \sin(x - t) + \cos(y - t),$$

to examine the convergence rates of Runge–Kutta methods of different orders. We employ the geographic mesh with h = L/64 and a quartic approximation. Errors and convergence rates are shown in Table 3, which demonstrate the correct orders of approximations for each of the different ordered Runge–Kutta methods.

6.1.2 Ambient Current Test Cases

To verify the wave model, we first test the propagation scheme in the presence of an ambient current (omitting source terms). We will examine four different cases of monochromatic, unidirectional waves, not necessarily normal to the coast, in the presence of different ambient currents. The first two cases examine an incoming wave traveling a distance of 4,000 m in deep water (H=10,000 m) in the presence of an opposing current (a) and a following current (b) with a velocity that increases from 0 to 2 m/s from the south to the north. Explicitly the current is

(a)
$$\mathbf{u} = \begin{pmatrix} 0 \\ -2/4,000y \end{pmatrix} \text{m/s}, \quad (b) \quad \mathbf{u} = \begin{pmatrix} 0 \\ 2/4,000y \end{pmatrix} \text{m/s}.$$

The incoming, monochromatic, long-crested waves are simulated with a Gaussian-shaped frequency spectrum, with peak frequency 0.1 Hz, standard deviation 0.01 Hz, and a $\cos^{500}(\theta)$



Fig. 3 Errors of the manufactured solutions

directional distribution (i.e., directional spreading is 5° [17]). The waves have a significant wave height of 1 m and a main direction of 90°. The incoming wave boundary is indicated with a blue line on the geographic mesh shown in Fig. 4. All other boundaries are absorbing. The spectral domain has 40 logarithmically distributed elements in frequency space that range from 0.05 to 0.25 Hz, which does not include the blocking frequency in the opposing current case, and has constant directional element spacing of $\Delta \theta = 2^\circ$. We are interested in the steady-state solution along the center of the geographic domain (the red line shown in Fig. 4). Steady-state solutions are achieved when there is little to no change in consecutive time-steps ($\Delta t = 2$ s) of the significant wave height in each geographic element. The results are shown in Fig. 5 curves (a) and (b). The DG solution is the black line and employs linears in both geographic and spectral space. We also show SWAN's stationary solution with the same numerical settings and also omitting source terms in blue for comparison because it is a well-trusted and widely-used wave model. The red line is the analytic solution (e.g.[15,22]), which is

	nRK=2,2		nRK=3,3		nRK=4,5			
Δt	L2 Error	β	L2 Error	β	L2 Error	β		
0.01	2.3971e-5	-	1.2205e-6	-	8.3565e-8	-		
0.005	2.3971e-5	0	1.2153e-7	3.3280	4.8452e-9	4.1083		
0.0025	5.9911e-6	2.0004	1.3692e-8	3.1500	-	_		
0.00125	1.4975e-6	2.0002	-	-	-	_		
0.000625	3.7437e-7	2.0001	_	_	_	_		

Table 3 Errors and convergence rates for different orders of Runge-Kutta methods

$$\frac{H_s^2}{H_{s,i}^2} = \frac{c_i^2}{c(c+2U)}$$

in which H_s is the wave height and c is the group velocity, where i represents the incident value. The DG solutions closely follow the analytic solutions. Note that SWAN produces significant wave heights that are larger than the analytic solution for the opposing current case. SWAN's solution improves slightly when we use $\Delta \theta = 0.2^{\circ}$ and 160 frequency elements.

The final two ambient current test cases turn the previous incoming waves by $\pm 30^{\circ}$ [for incoming main wave directions of 120° (c) and 60° (d)] in the presence of a slanted current,

$$\mathbf{u} = \begin{pmatrix} 2/4,\,000\,\mathrm{y}\\ 0 \end{pmatrix}\,\mathrm{m/s}.$$

The analytic solution (e.g. [12, 15]) is

$$H_{s} = H_{s,i} \sqrt{\frac{\sin(2\theta_{i})}{\sin(2\theta)}}$$
$$\theta = \arccos\left[\frac{gk_{i}\cos(\theta_{i})}{(\omega - Uk_{i}\cos(\theta_{i}))^{2}}\right].$$

Again we use the geographic mesh shown in Fig. 4. The spectral domain has 40 logarithmically distributed elements in frequency space that range from 0.05 to 0.25 Hz and has constant directional element spacing of $\Delta \theta = 2^{\circ}$ (c) or $\Delta \theta = 1.5^{\circ}$ (d). The steady-state solutions of the significant wave height and main wave direction are shown in Fig. 5 curves (c) and (d). The DG solutions using linears in geographic and spectral space are shown in black. SWAN is shown in blue and the analytic solution is shown in red. The DG solutions closely match the analytic solutions for both the significant wave height and the main direction demonstrating our model's ability to closely model incoming waves in the presence of an ambient current.

Further examining the opposing current case (a), we explore the benefit of using higher order approximations in geographic and spectral space and the effect of unstructured, geographic meshes on the DG solution. For this discussion we consider two coarse meshes, one with a structured and one with an unstructured, geographic mesh, and a fine mesh. The unstructured, coarse, geographic mesh, Fig. 6a, has similar resolution as the structured, coarse mesh, Fig. 6b. The structured, coarse geographic mesh has a sixteenth of the resolution of the geographic fine mesh, shown in Fig. 6c and the coarse spectral mesh only has about half the resolution as the fine spectral mesh ($\Delta\theta = 2^\circ$, 40 logarithmically distributed frequency



Fig. 4 The geographic mesh for the ambient current test cases. The *blue line* indicates the incoming wave boundary and the *red line* indicates the line along which the steady state solutions are shown



Fig. 5 Steady state solutions of the significant wave height, H_s , and the main direction, θ , for the ambient current test cases. The DG solution utilizing linears in both geographic and spectral space is compared to the unstructured SWAN and analytic solutions

elements). The steady state solutions of the significant wave height are shown in Fig. 7. For reference, SWAN solutions are shown in cyan and blue representing the structured, coarse and fine mesh solutions respectively. Note that the SWAN fine mesh solution is a significant



Fig. 6 The coarse unstructured (a) and structured meshes (b) and the fine mesh (c) used for the opposing current case in Fig. 7



Fig. 7 The steady-state solutions of the significant wave height for the opposing current case (a) are shown. We compare DG solutions with similar numbers of degrees of freedom, one on the fine mesh with constants and the other two on the coarse meshes, one with a structured and one with an unstructured geographic mesh, with linears in geographic space and cubics in spectral space. SWAN solutions on both the structured, coarse and fine meshes along with the analytic solution are also shown

improvement from the coarse mesh solution, which does not provide enough resolution. We show three DG solutions that employ either the structured or unstructured coarse mesh or the fine mesh but have similar numbers of degrees of freedom. The first, shown in green, employs the fine mesh with constant approximations in both geographic and spectral space (p,q=0). The second and third DG solutions, shown in magenta and black, employ the unstructured and structured coarse mesh, respectively, but with higher order approximations, linears in geographic space and cubics in spectral space (p=1,q=3). Comparing the SWAN fine mesh and the DG low-order, fine mesh solutions, we observe that SWAN's fine mesh solution is a better match to the analytic solution (shown in red). This is not unexpected because SWAN is using a higher order approximation than the constant approximation of the DG solution. Comparing the DG solutions we see a significant benefit in using higher orders of approximations on the coarse mesh versus a lower order of approximation on the fine mesh even though all solutions employ similar numbers of degrees of freedom. In particular, note that the coarse, higher-order solutions match the analytic solution, shown in red, better than the lower-order, fine mesh DG solution as well as SWAN's fine mesh solution. Also, note that the structure of the coarse mesh does not effect the solution.

6.1.3 Depth-Induced Shoaling and Refraction

We now test the propagation of monochromatic, long-crested waves in shallow water with varying depth and no current. We consider a wave propagating toward a plane beach over a distance of 4,000 m from a depth of 20 m (slope 1:200). The incoming wave has a significant wave height of 1 m, a Gaussian-shaped frequency spectrum, with peak frequency 0.1 Hz, standard deviation 0.01 Hz, and a $\cos^{500}(\theta)$ directional distribution. To test depth-induced shoaling we consider the incoming wave to have a main direction of 90° (a) and to test depth-induced refraction we consider the incoming wave to have a main direction of 120° (b). The analytic solution for both cases is given by (e.g. [21])

$$\frac{H_s^2}{H_{s,i}^2} = \frac{c_i}{c} \frac{\cos(\theta_i)}{\cos(\theta)}$$

and the wave direction is calculated with Snell's law. The geographic mesh is shown in Fig. 8; the element size *h* varies from 800 to 20 m as the depth decreases from 20 to 0 m (d = 20 - 1/200y). The spectral mesh has 40 frequency elements distributed logarithmically from 0.01 to 0.25 Hz and the directional elements have a constant spacing of $\Delta \theta = 5^{\circ}$ (SWAN solutions use $\Delta \theta = 0.25^{\circ}$). In Fig. 9, the steady-state solution of the significant wave height and main direction are shown where the x-axis corresponds to the red line indicated on the geographic mesh in Fig. 8. The DG solution employing linears in geographic and spectral space are shown in black and SWAN is shown in blue (for depths greater than 0.05 m). The DG model closely reproduces the analytic solution, shown in red, for the significant wave



Fig. 8 The geographic mesh for the depth-induced shoaling and refraction cases. The *blue line* indicates the incoming wave boundary and the *red line* indicates the line along which the steady state solutions are considered



Fig. 9 Steady state solutions of the significant wave height, H_s , and the main direction, θ , to the depthinduced shoaling and refraction cases. The DG solution utilizing linears in both geographic and spectral space is compared to the unstructured SWAN and analytic solutions

height and the turning of the main direction. Note the improved match to the analytic solution for wave directions, relative to SWAN.

6.2 Coupled Model

To verify the coupled DG wave/circulation model, we examine waves refracting over a circular shoal, as seen in Fig. 10, in the presence of a current. This test problem is similar to a test problem on structured meshes from Rogers et. al. [25] and Dietrich et. al. [9]. The source terms are neglected. At the north, south, and west boundaries, an incoming wave is prescribed by a JONSWAP (with peak enhancement factor $\gamma = 3$) spectrum with a significant wave height of 0.5 m, a peak period of 15.2 s, and a main direction of 335° with a cos¹⁴(θ) directional distribution (i.e., directional spreading is 15°). Incoming and outgoing fluxes are



Fig. 10 The bathymetry and geographic mesh for the circular shoal case

specified in DG-SWEM so that a current of 0.1 m/s flows from west to east. The geographic mesh with $h \approx 3,000$ m is shown in Fig. 10. The spectral domain has a directional spacing of $\Delta \theta = 10^{\circ}$ with 33 logarithmically distributed frequency elements that range from 0.05 to 1.0 Hz.

The DG wave/circulation model is coupled loosely. First, the circulation model is run, and it creates output files of the water levels and currents at every 600 s. These files are then used as input to the spectral wave model, which in turn outputs the radiation stresses every 600 s. These radiation stress files are then used as forcing in a new simulation of the circulation model. This process is repeated until there is no change in the output quantities.

For comparison purposes, we compare the loosely-coupled DG wave/circulation models with a tight coupling of DG-SWEM with SWAN. This tight coupling mimics the coupling of SWAN+ADCIRC as described in [9], so that the models run as the same executable on the same unstructured mesh, and information is passed through local memory without the need for interpolation. So DG-SWEM is either coupled loosely with the DG wave model, or tightly with SWAN. For both coupling paradigms, the inter-model communication occurs every 600 s. DG-SWEM uses a time step of 1 s, the DG spectral wave model uses a time step of 20 s and SWAN uses a time step of 600 s.

In Fig. 11, we show two DG coupled model solutions, one which uses constants (a) and another that uses linears (b) in the wave model in geographic space. Both use linear approximations in spectral space. Also shown in Fig. 11 are two SWAN+DG-SWEM solutions: the first is the solution on the original mesh (c) and the second is a fine grid "exact" solution (d). For the SWAN+DG-SWEM solution, we observe in Fig. 11 (c) that the waves refract prematurely when the shoal is represented on the original mesh. This is similar to the behavior observed in a test case without currents on structured meshes in [9], in which the authors showed that either refinement of the mesh or a limiter is needed to prevent the early refraction of the waves. The DG coupled model does not have premature refraction; however, the constant approximation (a) is too diffusive and linear approximations need to be used. The linear approximation (c) does qualitatively match the fine grid solution (d), again showing the benefit of higher orders in the DG method.



Fig. 11 The DG wave/circulation model results for the circular shoal are shown with constant (a) or linear (b) approximations in geographic space and linear approximations in spectral space in the wave model. SWAN's results tightly-coupled to DG-SWEM are also shown on the original mesh (c) and a fine mesh (d), which we consider 'truth'. The *black line* indicates the 110 and 290 m contours of the bathymetry

7 Conclusion

A spectral wave model has been developed to utilize the DG method in both geographic and spectral space. This numerical method allows for the use of unstructured geographic meshes and higher-order approximations in both geographic and spectral space. For the ambient current test cases as well as the depth induced shoaling and refraction test cases, we used linear approximations and accurately modeled the analytic solution of the significant wave height and the main wave direction. In addition, for the opposing current case, we showed that we obtain a more accurate solution by using higher-order approximations on a coarse mesh as opposed to a lower-order approximation on a refined mesh with similar numbers of degrees of freedom.

The DG spectral wave model has been loosely coupled to DG-SWEM. This coupled model employs the same unstructured geographic mesh, which eliminates interpolation error and can pass higher-order information between the two respective models. In the preliminary results of the coupled wave model, we found we needed a linear approximation in spectral space to properly resolve the significant wave height because the constant approximation was too diffusive. In general, we found that to obtain reliable results, for both the wave model individually and the coupled model, linear approximations are needed in spectral space; however, if larger spectral elements are used, even higher approximations might be necessary to maintain accuracy.

An a priori error estimate was performed for the formulated DG coupled wave/circulation model. The convergence rate of the model was found to be the minimum of p and q, the polynomial orders of approximation for geographic and spectral space respectively. Examining the wave model separately, we observed roughly p + 1 convergence rates when examining manufactured solutions for p = q and h small.

One potential drawback of the DG method is that it can be computationally more expensive than other methods such as finite element or finite difference methods due to the increased number of degrees of freedom. Going forward, we will optimize the DG spectral wave model for efficiency in addition to exploring *p*-adaptivity to obtain further speed-up. We will expand the capabilities to the DG spectral wave model, such as allowing for spherical coordinates and tightly coupling the wave model to DG-SWEM. Future work will also involve validating the model on realistic simulations with source terms.

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